# Parametric Resurgence and Large N Vortex Counting

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# Goals / Plan for talk

① Story of two halves:

- broad idea is to connect microstate counting + enumerative geometry.

② I will tell you about some work in 3d $\mathcal{N}=4$ gauge theory.

→ Lightning review.

→ Hemisphere blocks + quasimap counting

→ 3d ADHM with $\mathcal{M}_H = \mathrm{Hilb}^N \mathbb{C}^2$ and $AdS_4$


③ Why I became interested in resurgence.

→ It is fun

→ It is the natural place to study large N vpfns.

④ Resurgent trans-series with holomorphic parameters.

Interrupt me with qus!

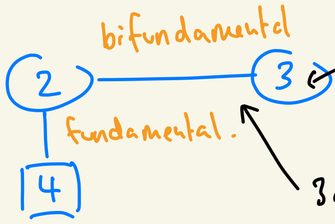
# Lightning Review of 3d $\mathcal{N}=4$ Gauge Theory

- Theory $\mathcal{T}$ preserving 8 supercharges $Q_\alpha^{a\dot{a}}$.

  → $SU(2)_E$ Lorentz. $SU(2)_H \times SU(2)_C$ R-symmetries.

- $\mathcal{T}$ is Lagrangian and specified by gauge group $G$ and quaternionic rep.

  → We will consider quiver gauge theories $\mathcal{T}_Q$ w. $R_Q = R \oplus \bar{R}$.



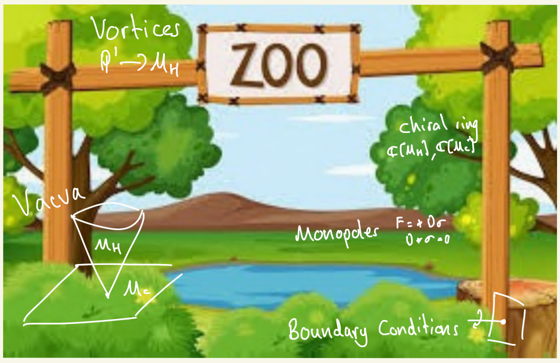

3d $\mathcal{N}=4$ vector $(A_\mu, \sigma, \varphi)$

3d $\mathcal{N}=4$ hyper $(X, Y)$

  → $\mathcal{T}_Q$ enjoys global flavour symmetry $G_H$ (and $G_C$)

  E.g. $(\mathbb{C}^\times)^4$

# BPS Zoo




Objects we can count or compute because they are supersymmetric = annihilated by some fraction of the $Q$s.

→ Today we will focus on vortices and vacua.

- <u>Supersymmetric vacua</u> annihilated by all $Q|0\rangle = 0$.
  $\mathcal{T}_Q$ has degenerate moduli space of vacua.

  → $\mathcal{M}_C$ - Coulomb branch where vector multiplet scalars attain a vev $(\varphi, \sigma)$
  Receives quantum corrections from monopoles

  → $\mathcal{M}_H$ - Higgs branch. Hypermultiplet scalars $(X, Y)$ attain vevs.
  Purely classical.
  Geometrically is the Nakajima quiver $\mathcal{M}_Q$ associated to $Q$

  $$\mathcal{M}_H = \mu^{-1}(0) \, /\!/ \, G_{\mathbb{C}}$$

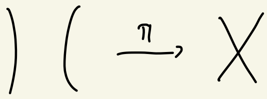
  → First order perspective of 3d mirror symmetry swaps $\mathcal{M}_H \rightleftarrows \mathcal{M}_C$
  $$\mathcal{T} \leftrightarrows \mathcal{T}^\vee$$

- EXAMPLE $\mathcal{T}_Q = SQED[N]$

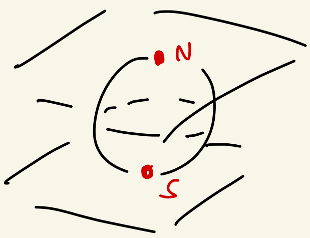
  (1) — [N]

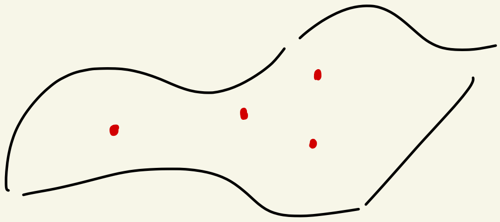
  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$. Global symmetry $G_H = (\mathbb{C}^\times)^N$ and R-symmetry $\mathbb{C}^\times_t$

  $(\mathcal{M}_C = \mathbb{C}[\text{monopoles}])$

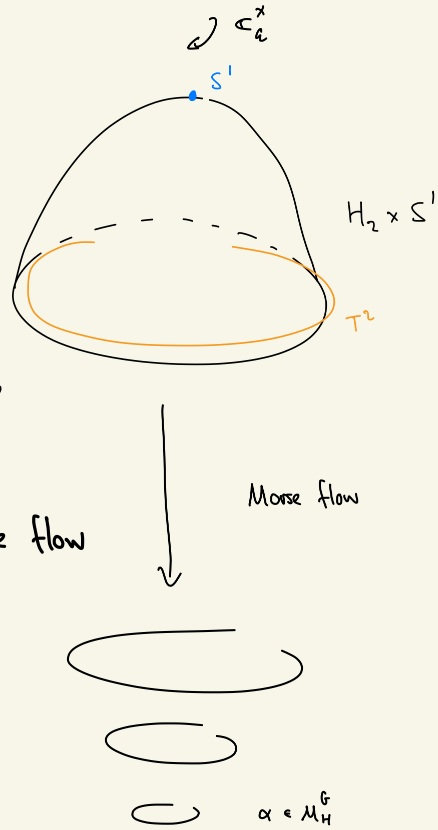



- We will turn on masses for $G_H$
  $m \in \mathfrak{g}_H$ sufficiently generic to lift $\mathcal{M}_H \to \mathcal{M}_H^{G_H}$



# Boundaries




- We will place $\mathcal{T}_Q$ on a hemisphere geometry.

- One may cut open 3-mfd ptn along a torus boundary.

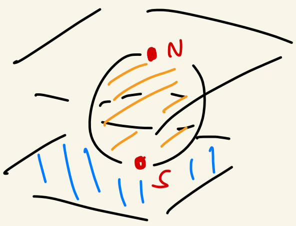
  The question is what boundary condition do we give at finite distance?

- Distinguished set of Lagrangian submanifolds $\mathcal{L}^+_\alpha \subset \mathcal{M}_H$ given by Morse flow w.r.t. mass parameters $m$.

  → We call this exceptional Dirichlet (there is a mirror 'Neumann' dual)

  Labelled by vacua $\mathcal{B}_\alpha$ w.r.t. $m$

EXAMPLE




# Hemisphere Block

- We compute the (flavoured) Witten index of states on $H_2$ w.r.t $\mathcal{B}^m_\alpha$

$$Z_\alpha(m, t, q) = \mathrm{tr}_{\mathcal{H}_{H_2}}(-1)^F \,(\text{flavours})$$

- Physical object - recipe for any $\mathcal{T}_Q$ purely in terms of Higgs branch data.

# Vortices

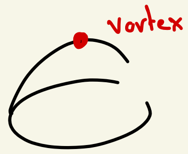
- One may compute the index by a path integral.

$$Z_\alpha = \int_{BCs} [d\varphi]\, e^{-S} \quad \underset{Q\varphi=0}{\rightsquigarrow} \quad \text{“}\sum \int_{Q\varphi \text{ moduli}}\text{”}$$

- We may choose a localisation scheme $Q$ s.t. $Q\varphi = 0$ are vortices.

- $Q$-cohomology described by: $\quad \vec{d} = \int_{\mathbb{P}^1} \mathrm{Tr}\, F \qquad X, Y \to \mathcal{L}^\dagger_\alpha$ at boundary

$$D_{\bar z} X = D_{\bar z} Y = 0$$




→ This data defines the quasimap moduli space

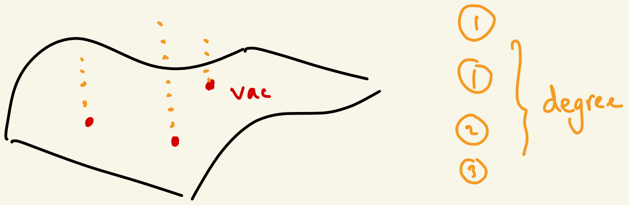
$$QM^d_\alpha = \{ f : \mathbb{P}^1 \to \mathcal{M}_H : f(\infty) = \alpha \}$$

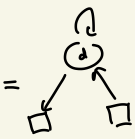
- The path integral is then computing K-theoretic intersection theory of $QM^d_\alpha$

  (= quantum K-theory of $\mathcal{M}_H$)

- In other words: $\quad Z_\alpha = \mathrm{tr}(-1)^F = \sum_d \chi_{G_H \times \mathbb{C}^\times_q}\left( \mathcal{O}^{vir}_{QM}, QM^d_\alpha \right).$

This may be further loclised to fixed quasimaps:




> Example $\quad \mathcal{T}_Q = SQED[N]$
>
> $\mathcal{M}_H = T^* \mathbb{P}^{N-1}. \qquad \widetilde{QM}^d_\alpha =$ 
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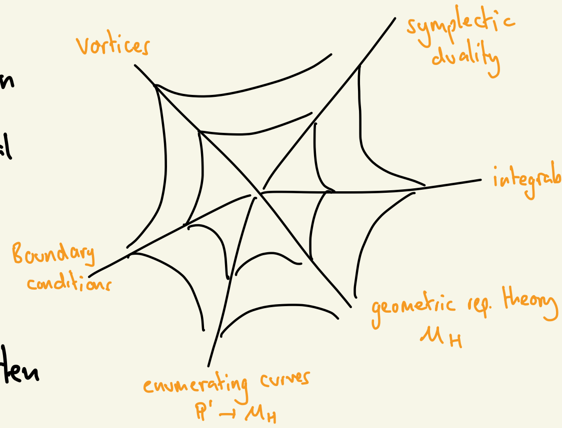
> $$Z_\alpha = \sum_{d \geq 0} \xi^d \prod_{i=1}^{N} \frac{(t x_i / x_\alpha ; q)_d}{(x_i / x_\alpha ; q)_d}.$$

- $\mathcal{T}_Q$ gives vast generalisation of $q$-hypergeometric functions. From now on think:

$$Z_\alpha = \sum_{\text{degree}} \xi^{\text{degree}} \sum f_{\text{rational}}(q, x = e^m, t)$$

# What is the point?

(Besides expanding the wonderful web of connections between 3d $\mathcal{N}=4$ theories and enumerative geometry.)




Our story is in contrast to a less rigorous approach to hd$^c$ factorisation

It is claimed that pfn on $\mathbb{R}^2_\varepsilon \times S^1$ can be expressed as a contour integral

- This amounts to the claim that $\chi(QM^d_\alpha, \mathcal{O})$ can be written

$$Z_\alpha = \sum_{\text{degree}} \xi^{\text{degree}} \sum f_{\text{rational}}(q, x = e^m, t)$$

$$= \sum_{\text{residues}} g(\text{residue}) = \int d(G\,\text{cartan})$$

However this is not always possible (such an expression is a Neumann BCs. and naive prescription needs to be modified.)

- Important when we consider factorisation $Z_{S^3} = \sum_\alpha Z_\alpha \bar{Z}_\alpha$

→ $q$ is an exact parameter. The supposed 'integrals' are evaluated as $q \to 1$:

$$Z_\alpha \sim \int dw\, e^{-\frac{1}{\varepsilon} S} + \dots = \sum_{\delta S = 0} \frac{1}{\sqrt{S''}} e^{-\frac{1}{\varepsilon} S_*}$$

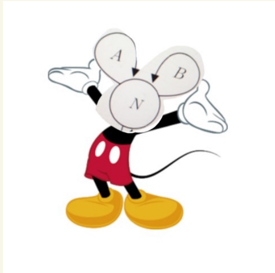
→ Actually this works but for other reasons:

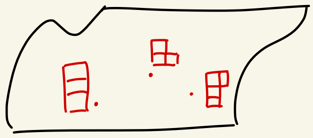
$\partial S = 0$ will always be Bethe eq$^n$'s of corresponding integrable system

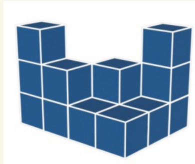
but there is not really a contour integral.

→ Why does this matter?

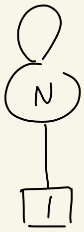
# The Hilbert Scheme







- Let us consider a special $\mathcal{T}_Q$ given by:




$$\mathcal{M}_H = \mathrm{Hilb}^N(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^N \mathbb{C}^2$$

Fixed points are $|\lambda| = N$ and vertices $(\Omega M_\lambda^T)$

are plane partitions. ($Z_\lambda$ 'counts' these)

- This theory flows in the IR to the worldvolume theory on a stack of N M2-branes.

  → It has an $AdS_4$ holographic dual.

  → Certain $AdS_4$ BH entropy can be recovered from the index $Z_\lambda = \mathrm{tr}(-1)^F$ when N (gauge group rank) is large.

- If one applies a naive integral saddle point approach one finds:

$$Z_\lambda = \int_{\Gamma_\lambda} dw\, e^{-S/\varepsilon} + \ldots \overset{\varepsilon\to 0}{=} \sum_\lambda \frac{1}{\sqrt{S''}}\, e^{-S_\lambda/\varepsilon} + \ldots \quad (q = e^{\varepsilon})$$

$$\overset{N\to\infty}{=} e^{\frac{N^{3/2}}{\varepsilon}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}} \quad \text{where } m = e^{\Delta} \text{ are global symmetries.}$$

- This is the correct entropy functional but lots of questions with this calculation. Three big ones are:

  (1) There is no well-defined integral for all $\lambda$. How to make sense of a large N saddle?

  → There are most likely more.

  (2) There is a trans-series structure $Z_\lambda = \sum \left(\frac{1}{N}\right)^i \varepsilon^j + e^{-1/\varepsilon} + e^{-N} + \ldots$

  → Can we decode this trans-series? Physics of subleading corrections?

  → Some conjectures that $\varepsilon$-terminates at large N.

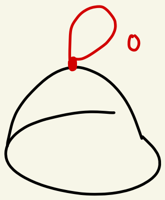
  (3) Is there interesting wall crossing / 'Stokes' phenomena as masses are varied.

- Resurgence is a tool to study such questions.

# Singular ODEs and lines

- Whilst there is no integral formula for $Z_\lambda$, there is an ODE at work that annihilates $P \cdot Z_\lambda = 0$.

- This arises as an action of charged Wilson lines at the tip of the hemisphere



$$O \cdot Z_\lambda = \partial_m \cdot Z_\lambda$$

- One can use the algebra of line operators $\{O_i\}$ to construct an action that annihilates $Z_\lambda$

$$P \cdot Z_\lambda = 0$$

- Geometrically these are relationships between enumerative counts w. K-theoretic descendants $\chi(QM, O_i)$
  → c.f. Witten conjecture. This is a kind of quiver variety version

EXAMPLE $\quad T_\varphi = SQED[N] \quad$ (a good toy model for $N$ and $\varepsilon$ limits - but no holography)

$$Z_\lambda = Z_\lambda(q, t, M, \varepsilon) \quad \text{and} \quad P = \prod_{i=1}^{N} (\varepsilon \partial_t + m_i) - \zeta^N$$

N.B. One can land on this theory by "Higgsing" a 5d theory.
$P$ is then the (quantised) SW/spectral curve of the 5d theory.

- We will use resurgence to analyse solutions to such singularly perturbed ODEs.
  → A full resurgent sol$^n$ is the next best thing to having an integral to perform steepest descent analysis.
  → Non standard however. $P$ is not single variable and depends on additional hol$^c$ parameters (masses)
  → I have spent a year in this resurgence rabbit hole. I have emerged and there is some really fun mathematics to talk about.

Holomorphic resurgence 🤝 3d $N=4$ gauge theory / symplectic duality, at large $N$.

# Divergent Series

- Warm-up example: $x^2 \frac{df}{dx} - f = -x$.

  → Seek a naive perturbative expansion $f_\infty(x) = f_0 + x f_1 + x^2 f_2 + \dots$

  → Will find $f_\infty(x) = \sum_{n \geq 0} n!\, x^n$ This is nonsense

  → Taylor's thm says $|f(x) - f_N(x)|$ is small for $N$ fixed
  $x$ small $\in \Delta \subset \mathbb{C}_x$

- Idea of Borel resummation is to assign an analytic value to $f_\infty(x)$.

  → First define the Borel transform $f_B(w) := \sum_{n \geq 0} \frac{f_n}{n!} w^n$. This converges.

  → Now recall $n! = \int_0^\infty dw\, w^n e^{-w}$. $f(w) := \int_0^\infty dw\, e^{-w/x} f_B(w)$
  has the asymptotics $f_\infty(x)$ but converges!

- This procedure is called Borel resummation.

Divergent $f_{a} = f_1 + x f_2 + x^2 f_3 + \dots$

Borel transform
$f_B(w) := \sum \frac{f_n}{n!} w^n$

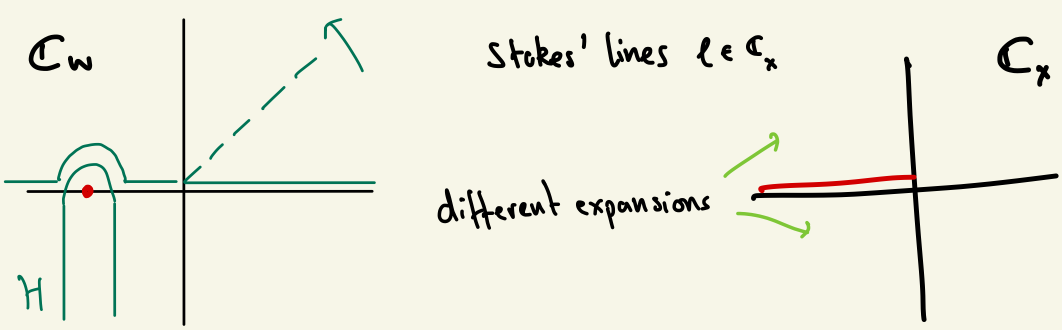
Borel sum
$f(x) = \int dw\, e^{-w/x} f_B$

- But, surprise! $f_B$ has a pole. In fact $f_B(w) = \dfrac{1}{1+w}$.

→ Consider the Borel sum $f(x) = \int_\gamma dw\, e^{-w/x} f_B(w)$.

$$\int_H dw\, e^{-w/x} \frac{1}{1+w} = e^{1/x}$$

Thus $f_a(x) \to f_a(x) + e^{1/x}$ across $\ell \in \mathbb{C}_x$




This is Stokes' phenomena.

# General Idea

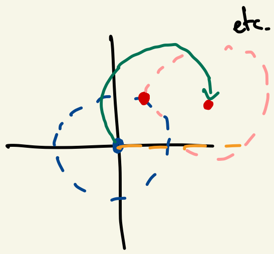
- Divergent perturbative series $f(x) \sim f_0 + f_1 x + f_2 x^2 + \dots$

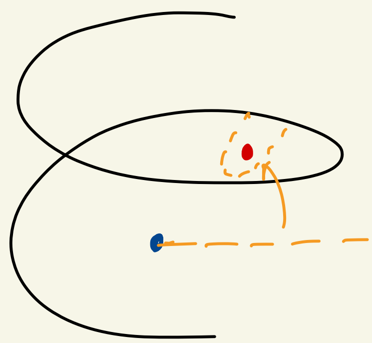
$\rightsquigarrow$ Holomorphic germ $f_B^0(w) = \frac{f_0}{0!} + \frac{f_1}{1!} w + \frac{f_2}{2!} w^2 + \dots$

- $f_B^0(w)$ may be analytically continued (Defines Riemann surface $\Sigma(f)$ )

→ Singularities of $\Sigma(f)$ determine non-perturbative contributions via $\int_{\Sigma(f)} d\lambda_x \sim f(x)$

- Stokes' lines occur when the contour snags on singularities.



# Resurgence 1/3

Idea : Extend algebraic perturbation theory from $c_0 + c_1 x + c_2 x^2 + \dots$ to include trans-monomials :

$$e^{-1/x},\ e^{-1/e^{-1/e^{-1/x}}},\ \log(\log x), \dots$$

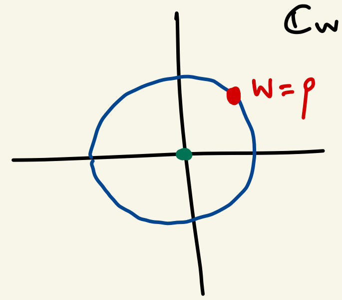
We will discuss height 1, log-free trans-series:

$$f(x) \sim c_0^{(0)} + c_1^{(0)} x + \dots + \sum_i e^{-A_i/x}\left(c_0^{(i)} + c_1^{(i)} x + \dots\right)$$

Ecalle's theory of resurgence is a correspondence:

$$\left\{ \begin{matrix} \text{Holomorphic functions} \\ \text{(resurgent functions)} \\ f_B \end{matrix} \right\} \xleftrightarrow[\text{Borel resummation}]{\mathcal{B}} \left\{ \begin{matrix} \text{(Divergent) trans-series} \\ \underbrace{c_0 + c_1 x + \dots + e^{-1/x}(c_0' + \dots)}_{f} \end{matrix} \right\}$$

# Resurgence 2/3




Recurring idea: (boring fact)

If $f_B(w) : \mathbb{C}_w \to \mathbb{C}_w$ is a holomorphic function with a single nearest singularity at $w = \rho$ of the form:

$$f_B(w) = (1 - w/\rho)^{-\alpha}\left(b_0 + b_1\left(1 - \frac{w}{\rho}\right) + \dots\right) + \text{reg.} \qquad \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$$

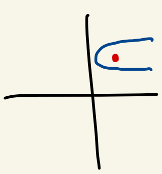
Then the coefficients in the local power series $f(w) = a_0 + a_1 w + a_2 w^2 + \dots$ enjoy the asymptotics:

$$\Gamma(n+1)\, a_n \sim \frac{\Gamma(n+\alpha)}{\rho^n}\left(\frac{b_0}{\Gamma(\alpha)} + \frac{\rho}{n+\alpha-1}\,\frac{b_1}{\Gamma(\alpha-1)} + \dots\right)$$

- Relatively mundane fact becomes remarkable when passed through the Ecallé correspondence:

$\rightsquigarrow$ $\{a_0, a_1, a_2, \dots\}$ some perturbative coefficients of $f_\infty(x)$.

$\rightsquigarrow$ $f(x) = \int_\gamma dw\, e^{-w/x} f_B(w)$, on a Stokes' line, the contour $\gamma$ snags on a singularity the perturbative series $f_\infty$ recieves an extra contribution from the Hankel contour:



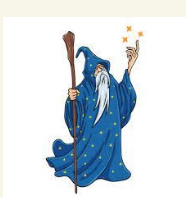
$$= \int_H dw\, e^{-w/x}\left(1 - \frac{w}{\rho}\right)^{-\alpha}\left(b_0 + b_1\left(1 - \frac{w}{\rho}\right) + \dots\right)$$

$\Gamma$-function

$$= \frac{2\pi i}{x^{\alpha+1}} e^{-\rho/x}\left(\frac{b_0}{\Gamma(\alpha)} + \frac{b_1 x}{\Gamma(\alpha-1)} + \dots\right)$$

Very strange: Feynman diagram expansions know about instantons.

- Hope to use (infinite dim. generalisations of) this idea to define non-perturbative QFT.

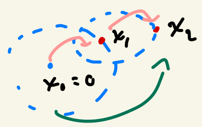


Divergent series know all! (not a surprise in $\mathbb{C}$!)

# Resurgence 3/3

- This structure persists (resurges!)

Holomorphic side

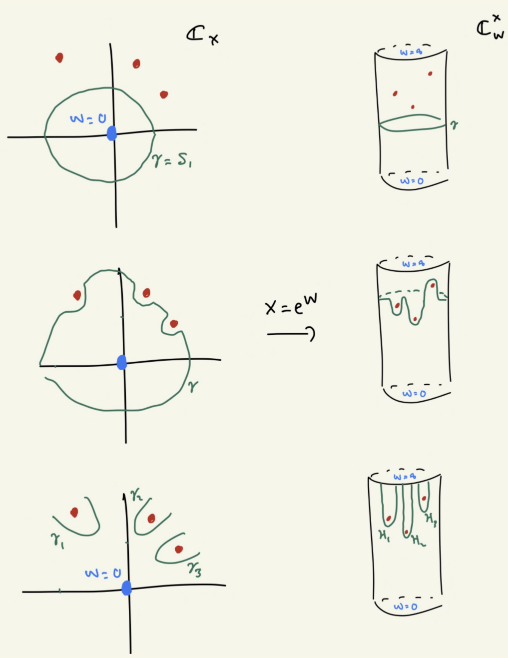



Perturbative side

$$y(x) = c_0^0 + c_1^0 x + \dots + e^{-x_1/\varepsilon}(c_0^1 + c_1^1 x + \dots) + e^{-x_n/\varepsilon}(c_0^2 + c_1^2 x + \dots)$$

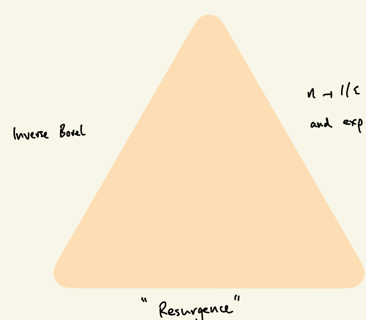
- This structure can be seen purely from the 1/n asymptotics of $c_n^{(0)}$.

→ In fact we show $c_B(w) = f_B(\varepsilon w)$




Trans-series

$$f(\varepsilon) \sim f_0^0 + f_1^0\varepsilon + f_2^0\varepsilon^2 + \dots + \frac{e^{-x/\varepsilon}}{\varepsilon^\alpha}(f_0^1 + f_1^1\varepsilon + \dots)$$




Borel singularities

$$f_B(w) \sim (w-x)^{-\alpha}(f_0^1 + f_1^1(w-x) + \dots)$$

Late-terms

$$f_n^0 \sim \frac{\Gamma(n+\alpha)}{x^{n+\alpha}}\left(f_0^1 + \frac{f_1^1}{n} + \dots\right)$$

# Upgrade!

- Our problem in enumerative geometry is of the general form: $P \cdot f = 0$ with

$$\text{"} \quad P = \left(\varepsilon \frac{d}{dt}\right)^N + g(m_1, \ldots, m_n) \quad \text{"}$$

  → We are interested in the asymptotic structure as $\varepsilon \to 0$

  → Compared with last slides we now have additional hol$^c$ variables $z = \{m, t\}$

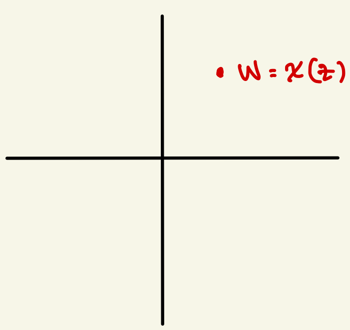
- Trans-series expansion now takes the form:

$$f(\varepsilon; z) = c_0(z) + \varepsilon c_1(z) + \varepsilon^2 c_2(z) + \ldots + e^{x(z)/\varepsilon}\left(c_0'(z) + \varepsilon c_1'(z) + \ldots\right)$$

  → We see a problem. What if $c_i(z)$ have singularities?

  → Classical idea of boundary layers – we explain this in resurgence context.

- The Borel germ $y_B(w; z) := \sum_{n=0} \frac{c_n(z)\, w^n}{n!}$ now depends on holomorphic parameters.




The analytic continuation now has two types of singularities

$$\left.\begin{array}{ll} \Gamma_z \subset \mathbb{C}_z & \text{'physical'} \\ \Gamma_w(z) \subset \mathbb{C}_w & \text{Borel.} \end{array}\right\} \text{mixed up}$$

Parametric Riemann surface $\Sigma_z$

- If we have a singular ODE e.g. $P = \varepsilon \frac{d}{dz} + G(z)$ then we may seek a solution in the Borel plane:

$$y(z,\varepsilon) = \int dw \, e^{-w/\varepsilon} \, y_B(w,z)$$

This yields a complex PDE in the Borel plane $P \to P_B = \partial_w + G(z)\partial_z$

- Introduce the idea of a 'template trans-series'.
  - → If we know the locations of Borel singularities $\chi(z) \in \Gamma_w$ and the local expansion then we recover the trans-series and Stokes' lines.
  - → We make a singularity ansatz $y_B(w,z) = (w - \chi(z))^{-\alpha} \left(c_0^i(z) + (w-\chi(z))c_1^i(z) + \dots\right)$
  - → Homogeneity of $P_B \Rightarrow$ obtain ODEs for trans-series components $c_i(z)$
  - → This is the template. How do we fill it with constants?

# Inner-Outer Matching

- Motto is a kind of 'Taylor expansion for trans-series'
- Where to set initial data for $\{ c_j^i(z) \}$? $\Gamma_z$ is a distinguished set of points in $\mathbb{C}_z$.
- Instead of expanding about $\Gamma_w(z)$ let us expand about $\Gamma_z$. For each $x(z)$ let $s = \frac{w}{x}$

  $\rightarrow$ $y_B(s, z) = z^{-\beta} \left( \varphi_0(s) + z\, \varphi_1(s) + z^2 \varphi_2(s) + \ldots \right)$

  $\rightarrow$ Form of the operator $P_B$ means $\varphi_i$ is coupled only to itself.

  " $\frac{d}{ds}\varphi_0 + \varphi_0 = 0$ ". Set of "constant" problems of type discussed previously.

- We may then compare the two expansions.

$$y_B(s, z) = (1-s)^{-\alpha} \left( a_0(z) + (1-s) a_1(z) + \ldots \right)$$

  $\rightarrow$ The resurgence lemma then gives a fun matching procedure for the $a_i(z)$ near $\Gamma_z$.

$$\begin{tikzcd} & \varphi_0(s) \arrow[dl] \arrow[d] \arrow[dr] & & & \varphi_1(s) \arrow[dl] \arrow[d] \arrow[dr] & & \cdots \\ a_0^0 & a_1^0 & a_2^0 \ \cdots & a_0^1 & a_1^1 & a_2^1 \ \cdots & \end{tikzcd}$$

- We now have all the ingredients of the trans-series.

- Q: What is the geometrical interpretation (QM counts) of the Borel PDE and this procedure?

  Q: Alien calculus?

- This structure applies to singularly perturbed PDEs $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$

  $\rightarrow$ $\Gamma_z \subset \mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$ has some geometry

  $\rightarrow$ $P_B$ propogates $\Gamma_z$ on characteristics $\mathbb{C}_z \times \mathbb{C}_w$

  $\rightarrow$ Inner-Outer matching zooms in. Recursive structure.

# Summary

- We define a physical object – the hemisphere block – with a finite distance boundary condition.

  Boundaries, Vermas and Factorisation – Bullimore, SC, Zhang

- For $Y_{\mathbb{C}} = \mathrm{Hilb}^N(\mathbb{C}^2)$ the entropy of certain $AdS_4$ BHs can be recovered but not from direct saddle point analysis – rather quantum toroidal integrability.

  Geometric Aspects of 3d $\mathcal{N}=4$ Gauge Theory – SC

- Resurgence is the natural tool to study the interplay of the Cardy $(q\to 1)$ and holographic $(N\to\infty)$ limits.

- We develop new tools to deal with additional $hol^c$ (mass parameter) dependence.

  Resurgent Aspects of Exponential Asymptotics – SC and Trinh

- It remains to put these pieces together and study resurgence of the quantum differential eq$^n$ of $\chi(QM_\lambda^d)$

Thank you! :)