Gauge theory, integrability and curve counting

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Based on hep-th/2010.09741 and ongoing work with M. Bullimore, H. Dinkins and D. Zhang.

Motivation and Background

- Extended algebras acting on BPS states of supersymmetric field theories in various dimensions
  - Supersymmetric indices/partition functions as characters
  - Quantum algebras acting on homology, K-theory, elliptic cohomology of quiver varieties
- 3d mirror symmetry/symplectic duality and enumerative geometry of quiver varieties evolve together with physics. Becoming indistinguishable. Physics <-> Geometry
- Exponential $N^{3/2}$ growth of states counted by indices. AdS_4 holography — saddle points. Geometry -> Physics?

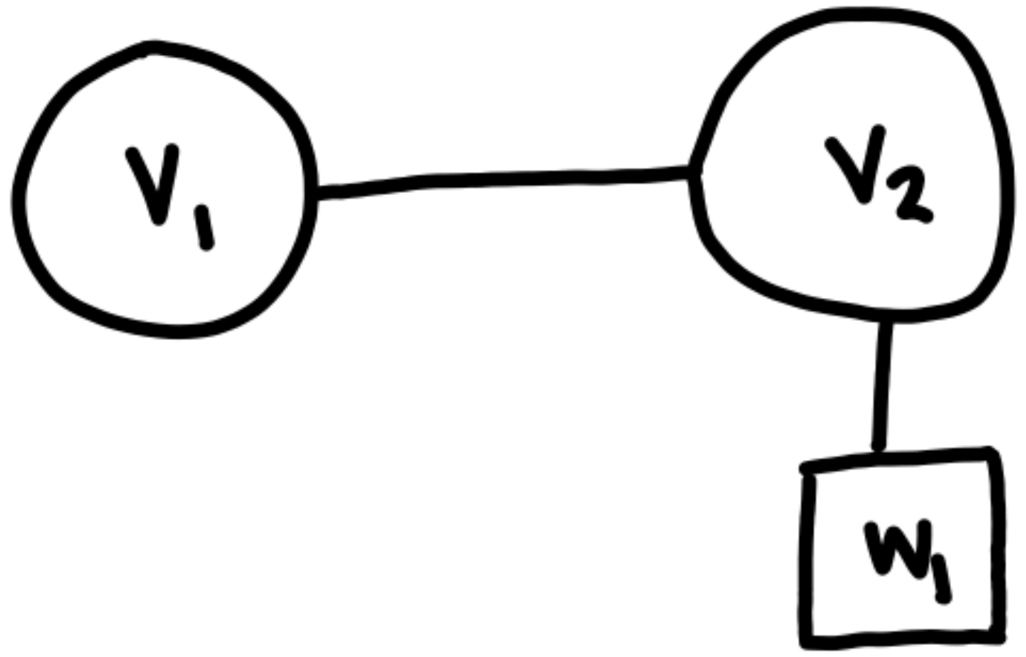
Outline

- Rapid review of (geometric aspects of) 3d $\mathscr{N} = 4$ (quiver) gauge theories
    - Vortices and curve counting
    - 3d mirror symmetry/symplectic duality
- Physical realisation of vertex functions and characters of quantised coordinate rings [Bullimore, SC, Zhang]
- Elliptic cohomology and duality interfaces [Bullimore, SC, Zhang (in preparation)]
- Mirror symmetry of twisted indices and quantum K theory of quiver varieties [Dinkins, SC, Zhang (in preparation)]
- Geometry of the AdS/CFT correspondence?

Background on 3d $\mathcal{N} = 4$ gauge theory

- SYM theory preserving 8 supercharges $Q_\alpha^{a\dot{a}}$
- Specified by gauge group G and representation $R \oplus \bar{R}$
- Built from vectormultiplets (A_μ, σ, φ) and hypermultiplets (X, Y)
- Symmetries:
  - $G_H \times G_C$ global symmetry
  - $SU(2)_H \times SU(2)_C$ R-symmetry

$\mathcal{T}_Q$



BPS Zoo

Monopoles

- 3d instantons $F = *D\sigma$, $D*\sigma = 0$
- Live in Q_C cohomology
- $H.S.[M_C] = \sum_m t^{\Delta(m)}$ [BFN, Bullimore et. al]

Vortices

- Time independent solitons
- Define stable maps $f: \mathbb{P}^1 \to M_H$

Vacua

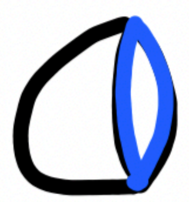
pair of symplectic resolutions

- Two branches M_H and M_C
- $N=4$ susy $\Rightarrow$ hyperkähler cones
- M_H is classical $(\mu^{-1}(0)//G)$
- $M_C^{cl.} = (\mathbb{R}^3 \times S^1)^{rk G} / W$

Chiral Rings

- $\mathbb{C}[M_H]$ and $\mathbb{C}[M_C]$ rings of local ops.

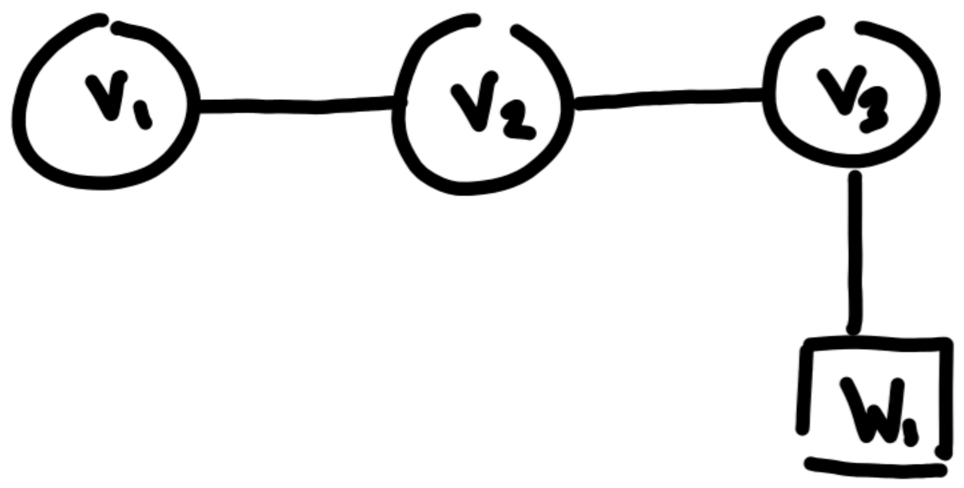
  G-invariant scalars   monopoles
- Captured by cohomology of Q_H and Q_C
- Graded by R-charge t. Equipped with bracket $\{,\}$.

Boundary Conditions 

- $N=(2,2)$ bcs. Dirichlet or Neumann
- Define Lagrangians $\mathcal{L} \subset M_H \times M_C$
- Associate with elliptic cohomology.

Quiver gauge theory

- Higgs vacuum moduli space $\mathscr{M}_H$ is a Nakajima quiver variety



FI parameters G_c

$$\mathcal{M}_H = T^*\mathrm{Rep}(\mathsf{v},\mathsf{w})/\!\!/_\theta G \qquad G_H \curvearrowright \mathcal{M}_H$$

real masses m

$$\mathbb{C}^\times_t \curvearrowright \mathcal{M}_H$$

R-symmetry
$U(1)_H \times U(1)_C$

$$\mathsf{T}_H = G_H \times \mathbb{C}^\times_t$$

Assume

- Generic FI parameters $\mathfrak{C}_C$ and masses $\mathfrak{C}_H$. $\mathscr{M}_H$ is smooth and $\mathscr{M}_H^{\mathsf{T}_H}$ is finite. $\alpha \in \mathscr{M}_H^{\mathsf{T}_H}$
- $\mathscr{M}_H$ is GKM.
- $\mathscr{T}$ is *good:* $\quad \mathrm{Tr}\,\mathbb{C}[\mathcal{M}_C] = \sum_{\mathfrak{m}\in\mathrm{Hom}(\mathbb{C}^\times,T)} t^{\Delta(\mathfrak{m};R)}$

$K_{\mathsf{T}_H}(\mathscr{M}_H)$ is generated by tautological classes $\mathscr{V}$ and $\mathscr{W}$

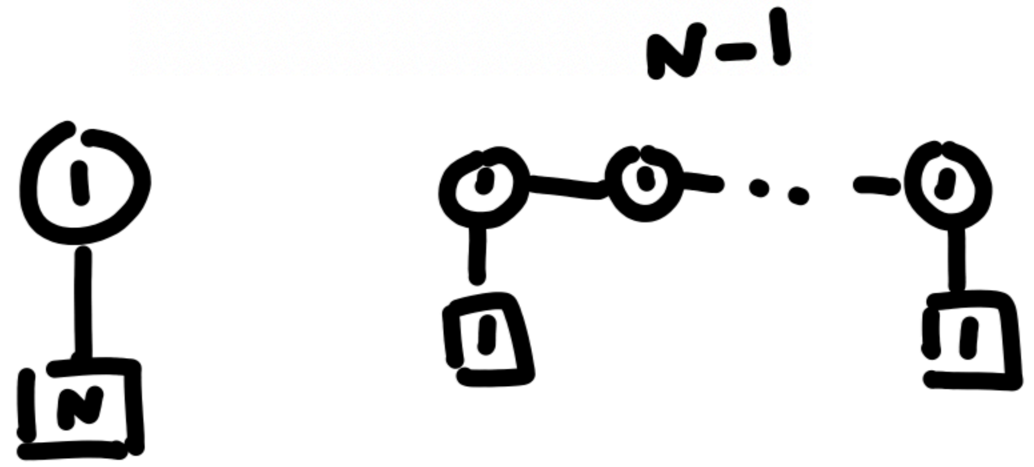
[McGerty and Nevins]

FI parameters $\zeta \in \mathrm{Pic}(\mathscr{M}_H) \times \mathbb{C}^\times$

$$K_{\mathsf{T}_H}(\mathcal{M}_H) = \mathbb{Z}[s_i^{\pm 1}, x_i^{\pm 1}, t^{\pm 1}]/R$$

3d mirror symmetry

E.g.



- Theories $\mathscr{T}$ also have a Coulomb branch $\mathscr{M}_C$
  - Parametrised by vevs of monopole operators built from vectormultiplet (A_μ, σ, φ)
  - Assume pair of symplectic resolutions $\mathscr{M}_H$ and $\mathscr{M}_C$

$$\mathcal{T} \longleftrightarrow \mathcal{T}^\vee$$

boundary conditions
vacuum moduli
chiral rings
solitons

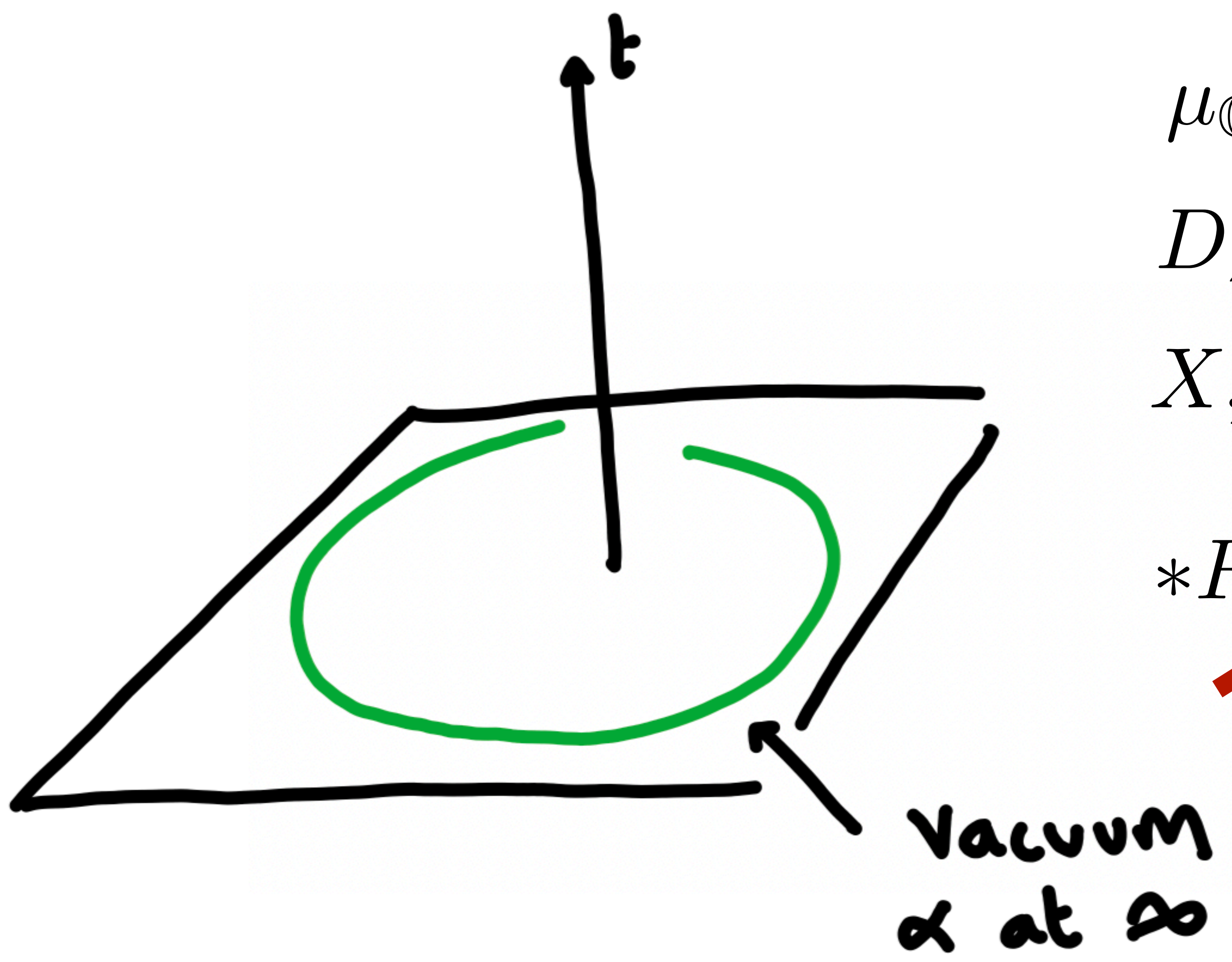
- 3d $\mathscr{N} = 4$ gauge theories come in mirror pairs [Intrilligator and Seiberg]
- Zeroth order statement:

$$\mathcal{M}_H(\mathcal{T}) \longleftrightarrow \mathcal{M}_C(\mathcal{T}^\vee)$$
$$\mathcal{M}_C(\mathcal{T}) \longleftrightarrow \mathcal{M}_H(\mathcal{T}^\vee)$$

- BPS objects match across the duality. We study boundary conditions and vortices.

Vortices and enumerative geometry

- Vortices are $1/2$ BPS time-independent solitons



$$\boldsymbol{d} = \int_{\mathbb{P}^1} \operatorname{Tr} F$$

$$\mu_{\mathbb{C}} = 0$$

$$D_{\bar{z}} X = D_{\bar{z}} Y = 0$$

$$X, Y \to \alpha \text{ at } \infty$$

$$\cancel{*F + \mu_{\mathbb{R}} = 0}$$

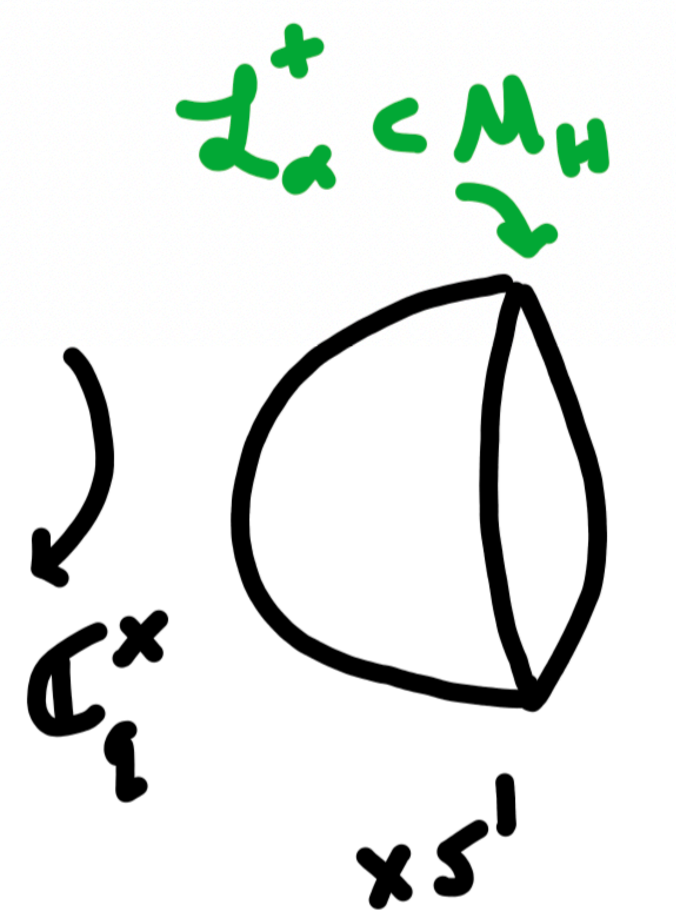
- [Okounkov et. al.] Quasimap moduli spaces

$$\mathrm{QM}^{\boldsymbol{d}}_{\alpha}(\mathcal{M}_H) = \{ f : \mathbb{P}^1 \to \mathcal{M}_H \ : \ f(\infty) = \alpha \,, \deg f = \boldsymbol{d} \}$$

- $\mathbb{C}^{\times}_q$ rotates base, T_H action on $\mathscr{M}_H$.

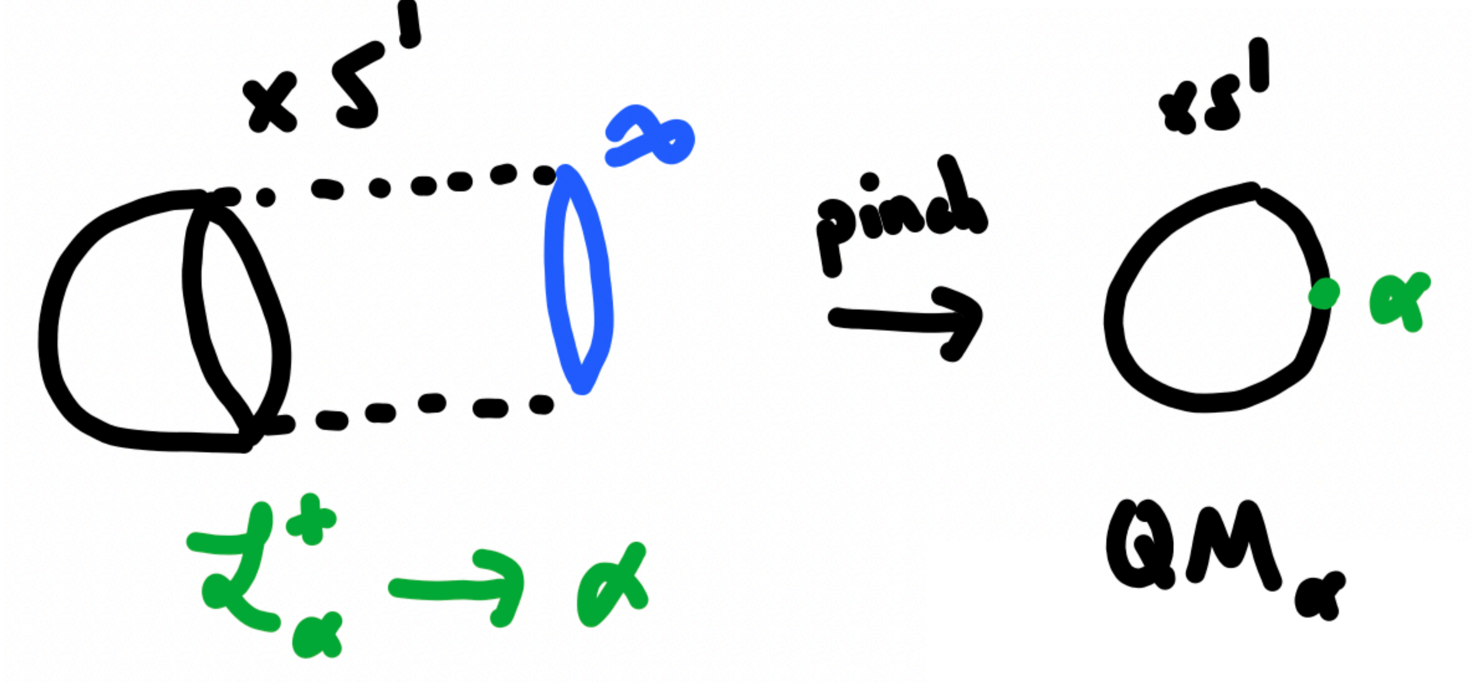
- Study invariants by equivariant localisation

Physical setup



- Place $\mathscr{T}$ on a hemisphere $S^1 \times H^2$
- Boundary condition $\mathscr{B}_\alpha$ at finite distance
- Particular class of (Dirichlet) boundary conditions associated to fixed chamber $\mathfrak{C}_H$

$$\begin{array}{c}\text{Exceptional} \\ \text{Dirichlet } B_\alpha\end{array} \leftrightarrow \begin{array}{c}\text{Lagrangians} \\ \mathcal{L}_\alpha^+ \subset M_H\end{array}$$



- We compute the partition function $\mathscr{Z}_{S^1\times H^2}(\mathscr{B}_\alpha)$ by supersymmetric localisation. (Vortex pfn).

$$\int_{M_\alpha} [dx]\, e^{-S} \xrightarrow[\text{Localise to } QM_\alpha^d(M_H)]{} \sum_d \int_{QM_\alpha^d} Z_{1\text{-loop}}$$

Vertex functions

equivariant curve count

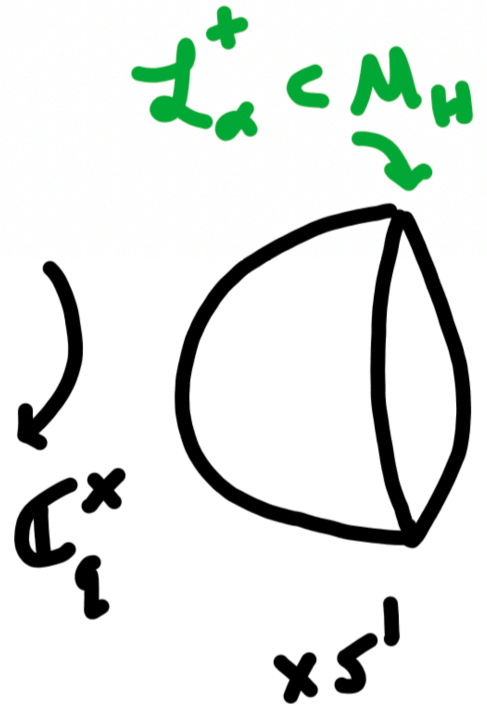
- *Vertex function* counts quasimaps $\mathbb{P}^1 \to \mathscr{M}_H$.

$$\mathsf{V}_\alpha(\mathcal{M}_H) \in K_{\mathbb{C}_q^\times \times \mathsf{T}_H}(\mathcal{M}_H)_{\mathrm{Loc.}} \qquad \mathsf{V}_\alpha(\mathcal{M}_H) = \sum_{\boldsymbol{d}\in H_2(\mathcal{M}_H,\mathbb{Z})} \zeta^{\boldsymbol{d}} \chi_{\mathbb{C}_q^\times \times \mathsf{T}_H}(\mathsf{QM}_\alpha^{\boldsymbol{d}})$$

- $\mathsf{QM}_\alpha^{\boldsymbol{d}}(\mathscr{M}_H)$ has a virtual deformation-obstruction theory. Compute the vertex function by localisation.

- Upshot:

$$\mathcal{Z}_{S^1\times H^2}(\mathcal{B}_\alpha) = e^{\phi_\alpha}\,\mathrm{PE}\left[\frac{1-t}{1-q}N_\alpha^+\right]\mathsf{V}_\alpha(x,\zeta;q,t)$$



$$\phi_\alpha = \sum_{i\in I} \xi_i \, \mathcal{L}^{(i)}\Big|_\alpha$$

State-operator map

- Place theory on $\mathbb{R}^{\geq 0} \times \mathbb{R}^2$ with Ω-deformation. $Q^2_{H,C} = \mathscr{L}_V$. `Alternative quantisation'.
- Quantises the bulk rings $\mathbb{C}_q[\mathscr{M}_{H/C}]$ — quantum symplectic reduction, classified by $H^2(\mathscr{M}_{H/C}, \mathbb{C})$
- State-operator principle tells us: $\mathcal{H}_{H^2} = \mathrm{Ops}_{\mathbb{R}^2_0}$
- In particular, sum over boundary monopoles.

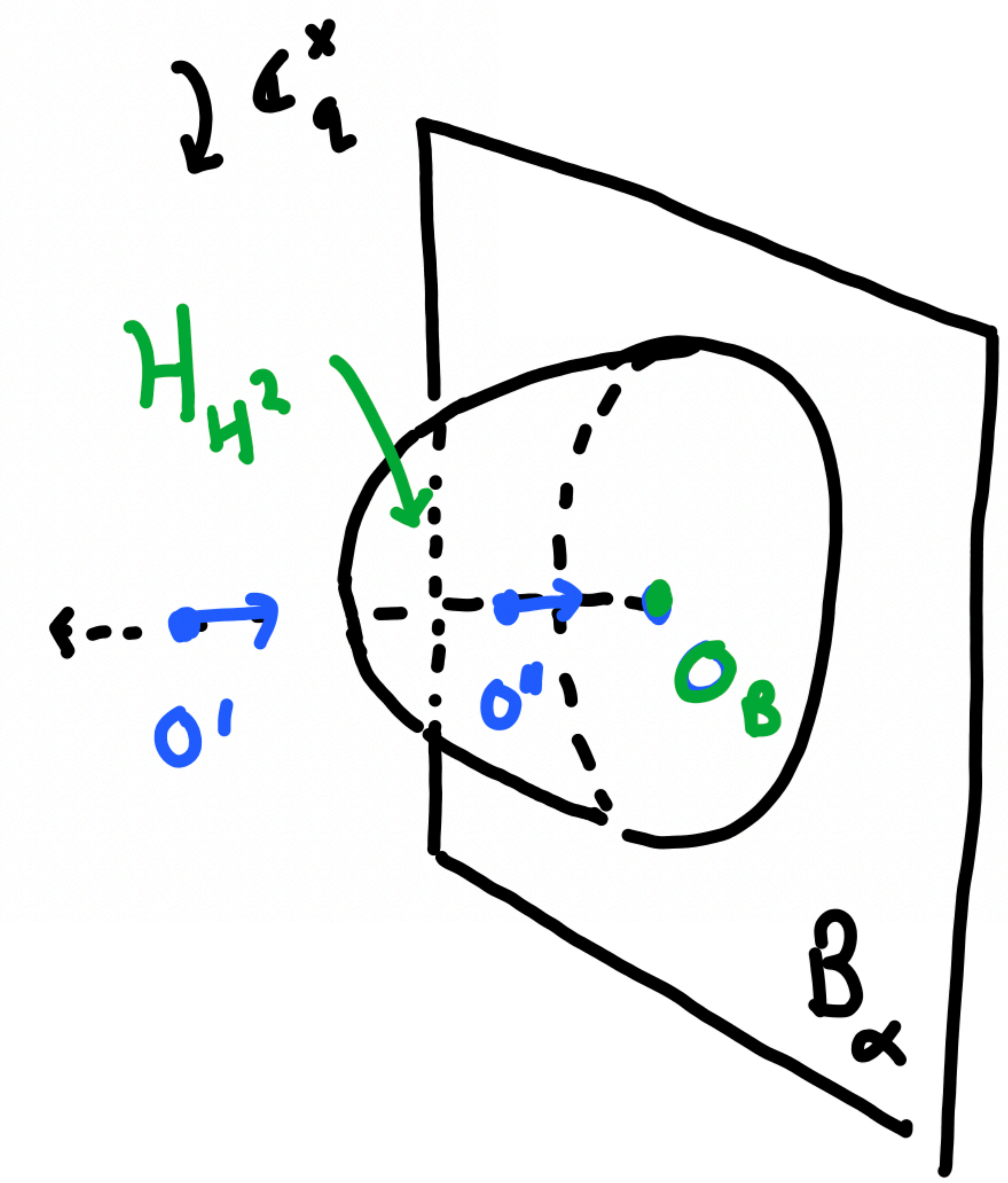
half index

$$\mathcal{Z}_{S^1 \times H^2} \sim \mathrm{Tr}_{H^2}\, (-1)^F e^{\{Q,Q^\dagger\}} = \mathrm{Tr}_{\mathrm{Ops}}\, (-1)^F e^{\{Q,Q^\dagger\}}$$

- Heuristically:

states — operators

$$\int_{QM_\alpha} \hat{A}(T_{vir.}) = \sum_{f.p.} \hat{a}(T_{f.p.} QM)$$



Example

$\mathcal{M}_H = T^*\mathbb{P}^{N-1}$

$\sim$

$\sim$

$K_T(\mathcal{M}_H) = \mathbb{Z}[s^{\pm 1}, x_i^{\pm 1}, t] \,/\, R \qquad \nu|_\alpha : s \mapsto x_\alpha^{-1}$

$TX = s(x_1 + \dots + x_N) + t s^{-1}(x_1^{-1} + \dots + x_N^{-1}) - 1 - t$

$N_\alpha^+ = \sum_{i<\alpha} s x_i + t \sum_{i>\alpha} s^{-1} x_i^{-1}$

$G_H = (\mathbb{C}^\times)^N \quad x_1, \dots, x_N \qquad R_H = \mathbb{C}_t^\times$

Fix chamber $\mathfrak{C}_H = \{x_1 < \dots < x_N\}$

vacua are $\alpha = 1, \dots, N$

$\mathcal{L}_\alpha^+ =$ conormal bundle to schubert cells $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{\mathbb{P}^i}(-1)$

$$\mathcal{I}(B_\alpha) = \sum_d \xi^d \, PE\left[\frac{1-t}{1-q} N_\alpha^+\right] \Big|_{s = x_\alpha^{-1} q^d}$$

boundary monopoles

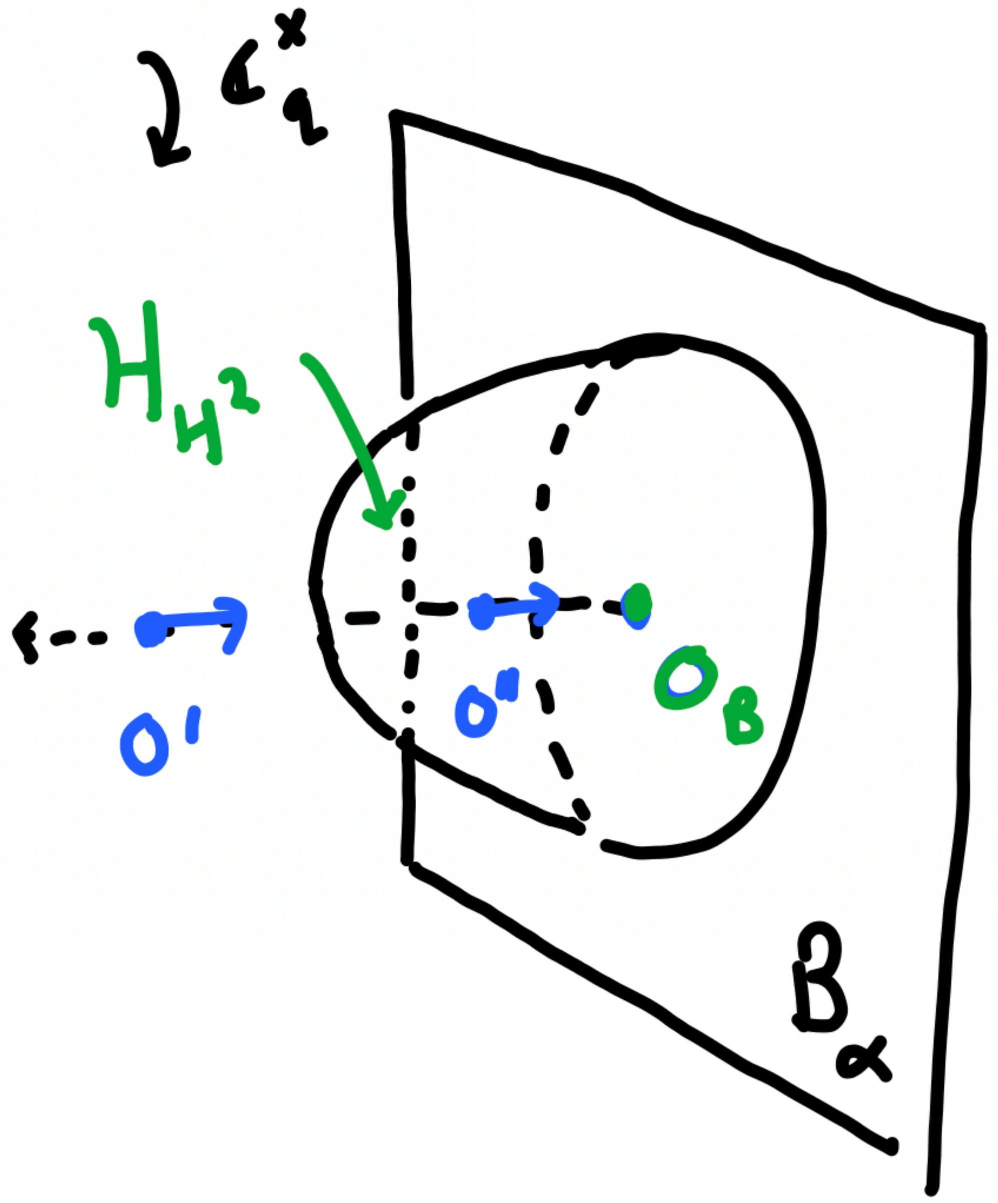
$$= PE\left[\frac{1-t}{1-q} N_\alpha^+\right] \sum_d \xi^d \prod_{i=1}^{N} \frac{(t q x_i / x_\alpha ; q)_d}{(q x_i / x_\alpha ; q)_d}$$

$\chi(QM_\alpha^d, \hat{\mathcal{O}}_{vir.})$

Exceptional Dirichlet

$B_\alpha \sim \mathcal{L}^+_\alpha \subset M_H$

Modules



- Half index: $\mathcal{I}(\mathcal{B}_\alpha) = \mathrm{Tr}_{\mathrm{Ops}}\,(-1)^F e^{\{Q,Q^\dagger\}}$

- In *specialised limits* (A-limit and B-limit) the index receives contributions from only Q_H or Q_C cohomology. Higgs or Coulomb operators for good theories.

- Geometrically (states), counts holomorphic functions supported on Lagrangians or fixed points of QM moduli space.

- Algebraically (operators), half index of $\mathscr{B}_\alpha$ gives Verma character of $\mathbb{C}_q[\mathscr{M}_H]$ or $\mathbb{C}_q[\mathscr{M}_C]$.

$$\mathcal{Z}_{S^1\times H^2}(\mathcal{B}_\alpha) = e^{\phi_\alpha}\mathrm{PE}\left[\frac{1-t}{1-q}N_\alpha^+\right]\mathrm{V}_\alpha(x,\zeta;q,t)$$

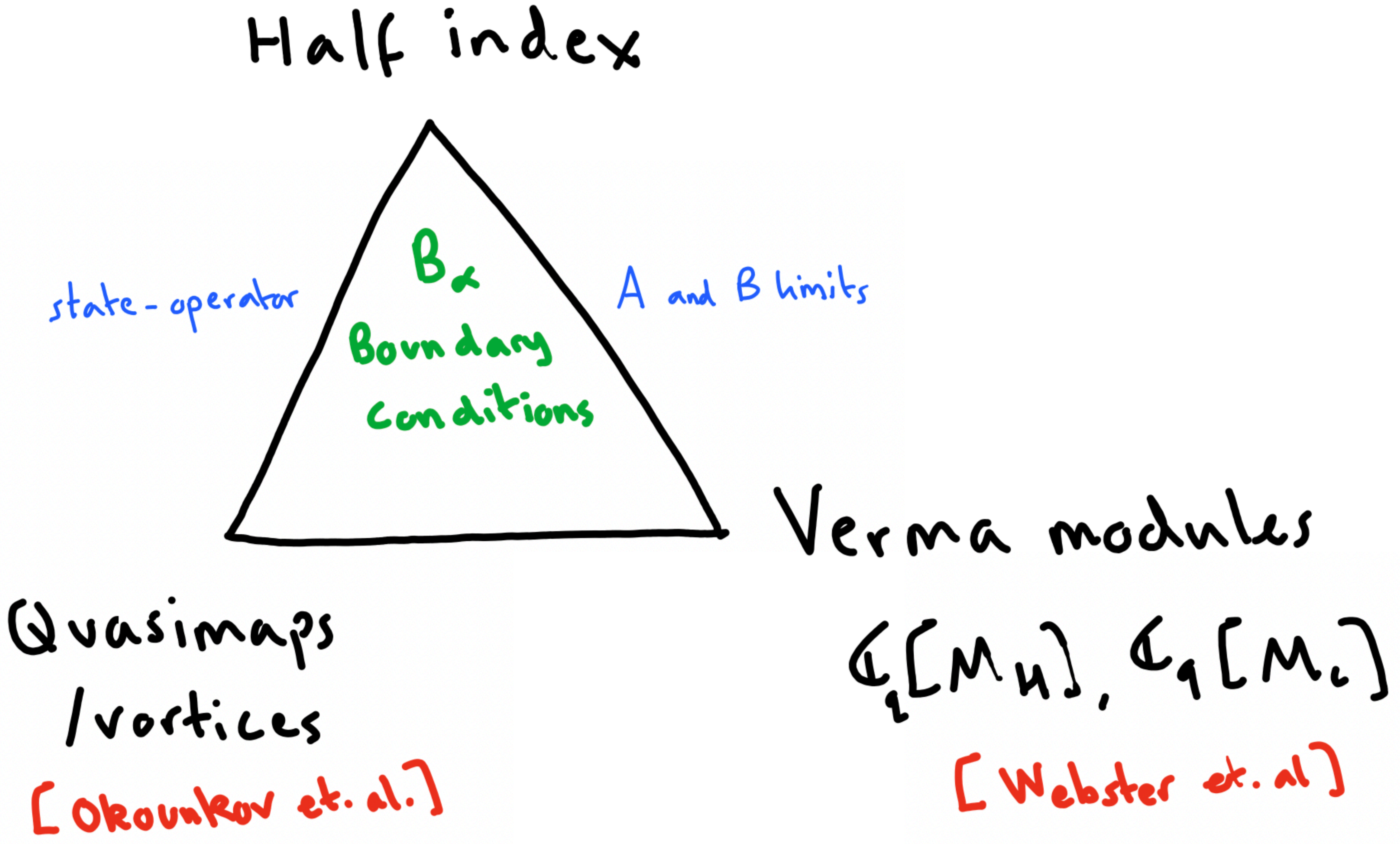
cl. 1-loop non-pert.

A $\nearrow$ $Z^A_\alpha = e^{\varphi^A_\alpha}\sum_{\text{f.p QM}^d}\zeta^d$

B $\searrow$ $Z^B_\alpha = e^{\varphi^B_\alpha}\,\mathrm{PE}[N^+_\alpha]$

$$= \mathrm{PE}\left[\frac{1-t}{1-q}N^+_\alpha\right]\sum_d \zeta^d \prod_{i=1}^{N}\frac{(t q x_i/x_\alpha\,;q)_d}{(q x_i/x_\alpha\,;q)_d}$$

$\chi(\mathrm{QM}^d_\alpha, \hat{\mathcal{O}}_{\mathrm{vir.}})$



Half index
state-operator
A and B limits
B_α
Boundary conditions
Verma modules
$\mathbb{C}_q[M_H]$, $\mathbb{C}_q[M_C]$
[Webster et. al]
Quasimaps /vortices
[Okounkov et. al.]

Elliptic cohomology

- K-theory class: $|\mathscr{M}_H^{\mathsf{T}_H}|$ copies of functions on $(\mathbb{C}^\times)^N$ $\quad K_{\mathsf{T}_H}(\mathcal{M}_H) \hookrightarrow \bigoplus_{\alpha \in \mathcal{M}_H^{\mathsf{T}_H}} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$

- Elliptic cohomology class: $|\mathscr{M}_H^{\mathsf{T}_H}|$ copies of functions on $(E_\tau)^N$ $\quad E_{\mathsf{T}_H}(\mathcal{M}_H) \hookrightarrow \bigoplus_{\alpha \in \mathcal{M}_H^{\mathsf{T}_H}}$ 'functions' on E_τ^N

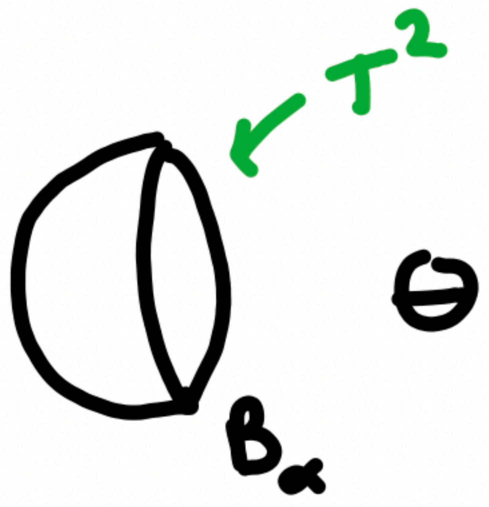
- (Extended) elliptic cohomology classes are sections of line bundles over the elliptic cohomology scheme

$$E_{\mathsf{T}_H}(\mathcal{M}_H) := \left(\bigsqcup_{\alpha \in \mathcal{M}_H^{\mathsf{T}_H}} \mathcal{O}_\alpha \right) / \Delta \qquad \mathcal{O}_\alpha = E_\tau^N \times E_\tau^{\mathrm{Pic}(\mathcal{M}_H)}$$

GKM: $s_\alpha|_{O_\alpha \cap O_\beta} = s_\beta|_{O_\alpha \cap O_\beta}$
if α, β joined by curve C

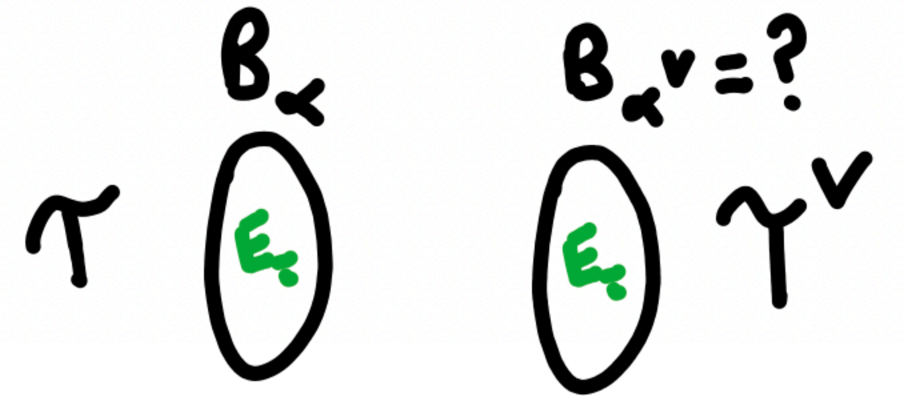
- Notion of fundamental class $\quad \Theta : K_{\mathsf{T}_H}(\mathcal{M}_H) \to E_{\mathsf{T}_H}(\mathcal{M}_H)$

Elliptic cohomology classes $\leftrightarrow$ Partition functions of $N=(2,2)$ b.c.s.

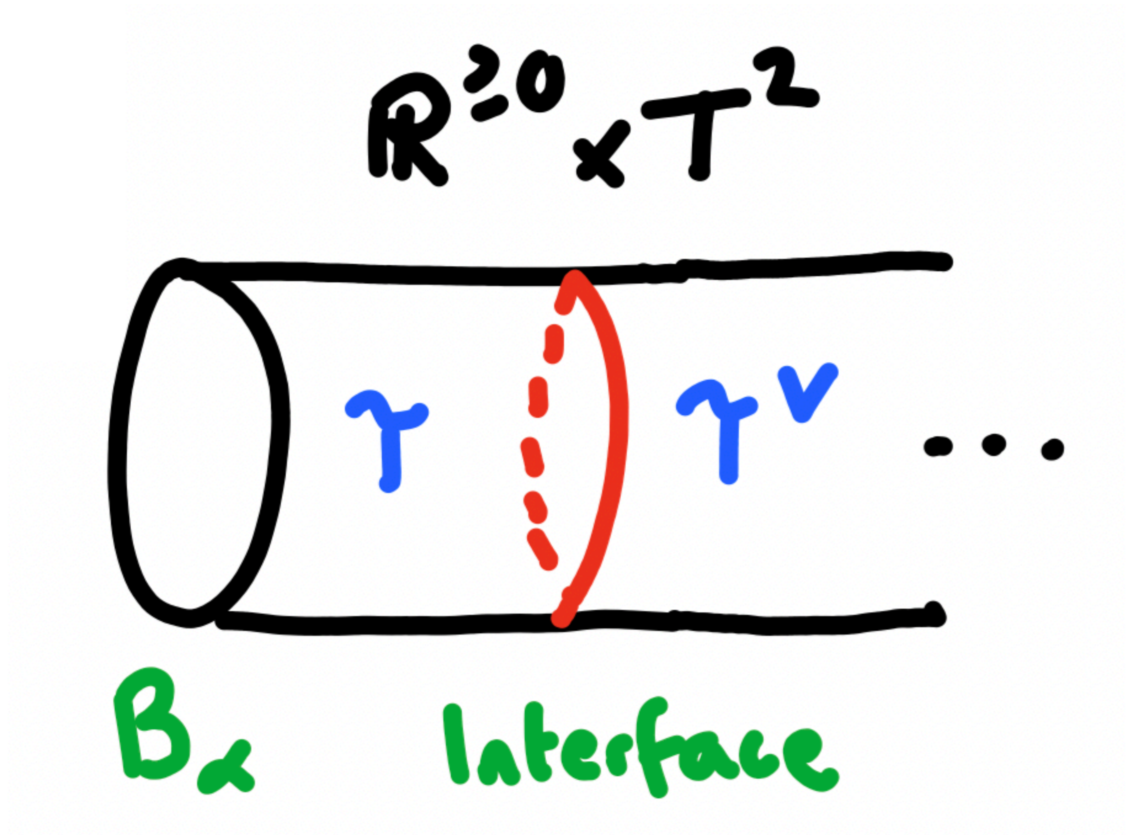


$\Theta(\mathcal{L}_\alpha^+)$

Mirror dual boundary conditions



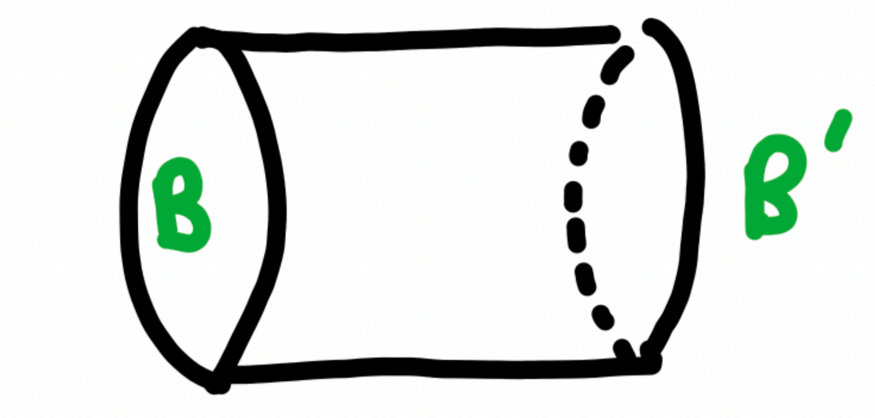
- Mirror symmetry should exchange vertex functions. What is the dual boundary condition?



- Collide exceptional Dirichlet with interface. $\mathcal{I} \in E_{\mathsf{T}_H \times \mathsf{T}_C}(\mathcal{M}_H \times \mathcal{M}_C)$

- Collision = putative inner product on elliptic cohomology.

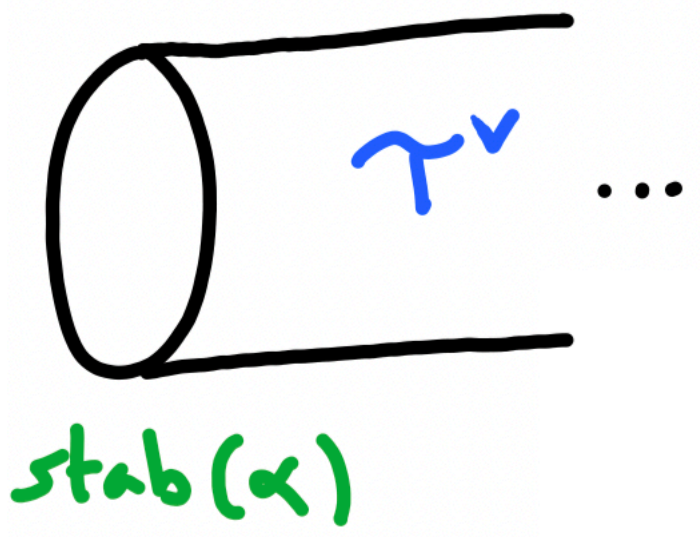
$Z_{I \times T^2} = \langle B, B' \rangle$



c.f. K-theory

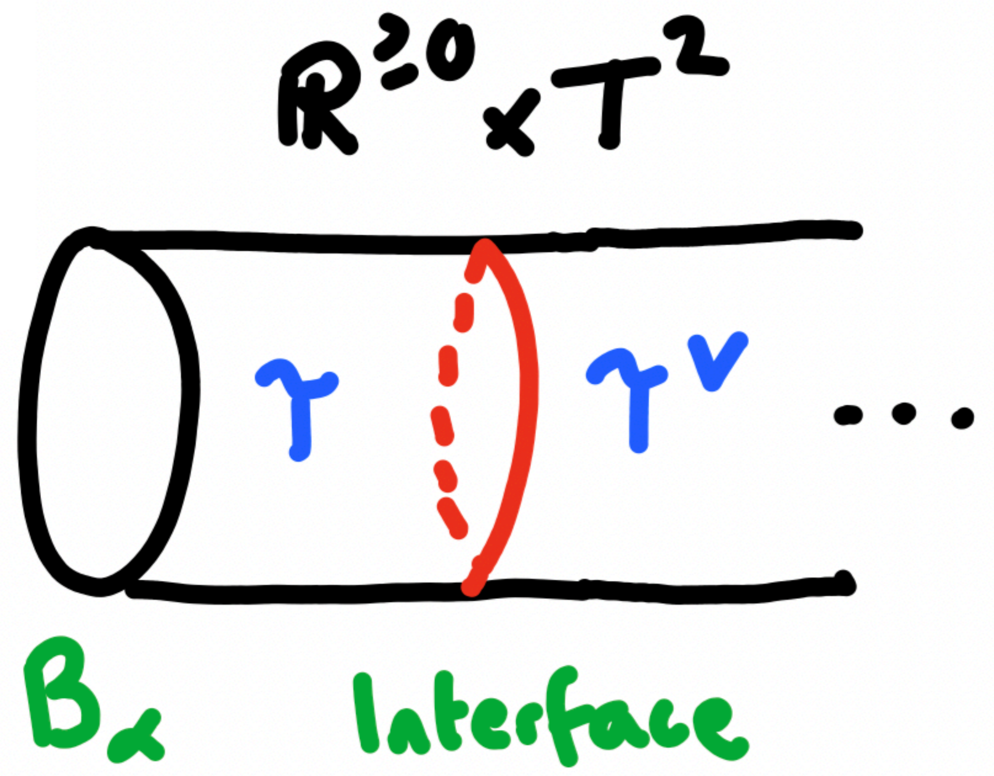
$\langle B, B' \rangle = \sum_\alpha B_\alpha B'_\alpha \, PE[TM_H] = \chi(B \otimes B'; M_H)$

- We obtain the elliptic stable envelope of [Aganagic, Okounkov]



$$\langle \mathcal{B}_\alpha, \mathcal{I} \rangle = \mathrm{Stab}_\alpha^{\mathrm{Elliptic}}$$

Mirror symmetry of vertex functions



- Elliptic stable envelope is a map $\mathrm{Stab}: E_{\mathsf{T}_H}(\mathcal{M}_H^{\mathsf{T}_H}) \to E_{\mathsf{T}_H}(\mathcal{M}_H)$

- The support is *stable* $\mathrm{Supp}\,\mathrm{Stab}(\alpha) \subset \bigcup_{\beta \leq \alpha} \mathcal{L}^+_\beta$

- 'Up-down' basis in spin chain

- Collision gives the dual boundary condition as *enriched Neumann*. The Higgs branch image is the stable envelope.

$$\mathcal{I}(\Upsilon; B_\alpha) = \mathcal{I}(\Upsilon^\vee; B^\vee_{\alpha^\vee})$$

Vertex fct. of $\mathcal{M}_H$

$$= \int_{JK} \frac{ds}{s}\, \mathrm{Stab}^E(\alpha)\, [\text{hyper. contributions}]$$

$$= \sum_{\beta \leq \alpha} \mathrm{Stab}^E(\alpha)\big|_\beta\, \mathcal{I}(\Upsilon^\vee; B_{\beta^\vee})$$

Vertex function of $\mathcal{M}_C$

Example:

$\mathcal{T} =$  $, \ \mathcal{M}_H = \mathrm{Hilb}^N(\mathbb{C}^2)$

$T_H = \mathbb{C}^\times_z \times \mathbb{C}^\times_t \qquad \lambda =$ 

$K_T(X) = \mathbb{Z}[\,w_i^{\pm 1},\ z^{\pm 1} t^{\pm 1}]/R$

$w_i = z^{a_\lambda(i)} t^{l_\lambda(i)}$

$$\mathcal{I}^{B,\lambda}_{\text{1-loop}} = \mathrm{PE}\left[\frac{t-q}{1-q}N^+_\lambda\right] = \prod_{s\in\lambda}\frac{\left(qz^{a_\lambda(s)+l_\lambda(s)+1}t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)};q\right)_\infty}{\left(tz^{a_\lambda(s)+l_\lambda(s)+1}t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)};q\right)_\infty}$$

$$\mathrm{V}_\lambda(\zeta,z;q,t)$$
$$= \sum_{\pi\in\mathrm{RPP}(\lambda)}\left(\zeta t^{-\frac{1}{2}}q^{\frac{1}{2}}\right)^{|\pi|}\prod_{s\in\lambda}\frac{(w_s(\lambda)^{-1};q)_{-\pi_s}}{(qtw_s(\lambda)^{-1};q)_{-\pi_s}}\prod_{\substack{s,t\in\lambda\\ s\neq t}}\frac{\left(qt\frac{w_t(\lambda)}{w_s(\lambda)};q\right)_{\pi_t-\pi_s}}{\left(\frac{w_t(\lambda)}{w_s(\lambda)};q\right)_{\pi_t-\pi_s}}\frac{\left(zt^{-\frac{1}{2}}\frac{w_t(\lambda)}{w_s(\lambda)};q\right)_{\pi_t-\pi_s}}{\left(qzt^{\frac{1}{2}}\frac{w_t(\lambda)}{w_s(\lambda)};q\right)_{\pi_t-\pi_s}}$$

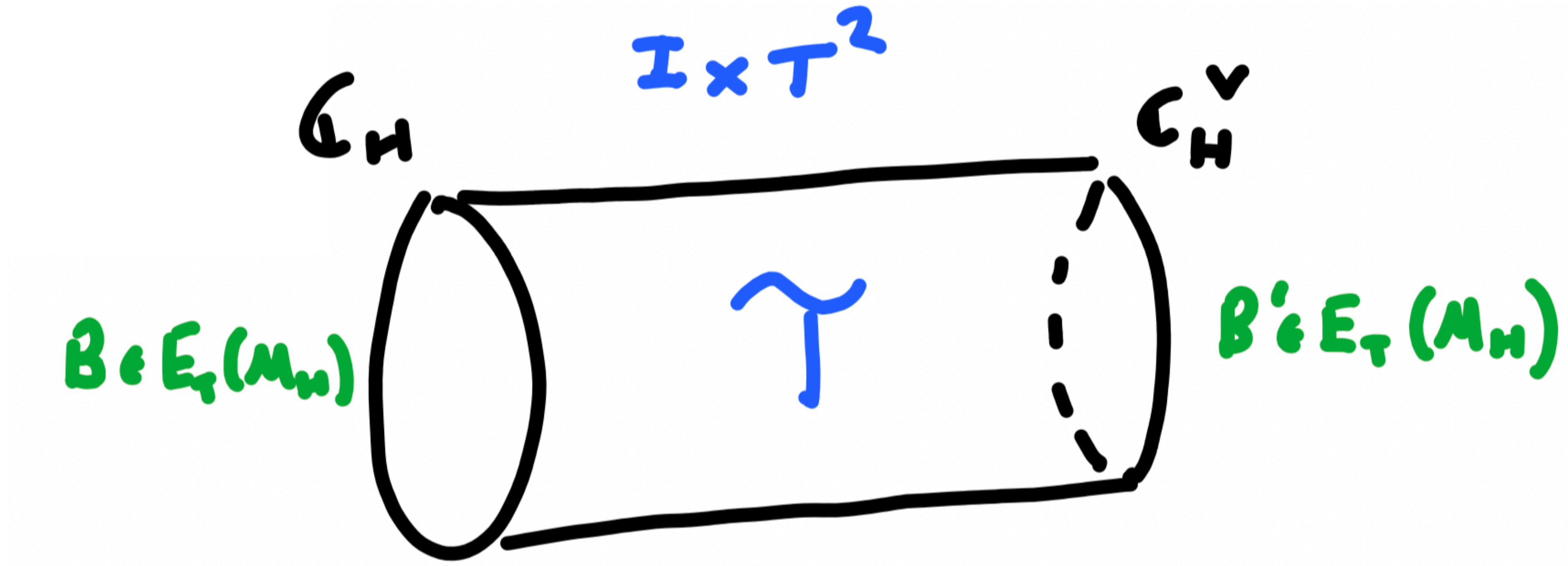
- Elliptic stable envelope becomes diagonal in specialised limits:

$$\lim_B \mathcal{I} = \prod_{s\in\lambda}\frac{1}{1-z^{h_\lambda(s)}} = \lim_A \mathcal{I} = \sum_{\pi\in\mathrm{RPP}(\lambda)} \zeta^{|\pi|}$$

Verma characters of $\mathbb{C}_q[\mathrm{Hilb}^N(\mathbb{C}^2)]$?

[Nakajima + Kodera]

Unitarity of Stab



- Left and right means flip chamber (reverse orientation)

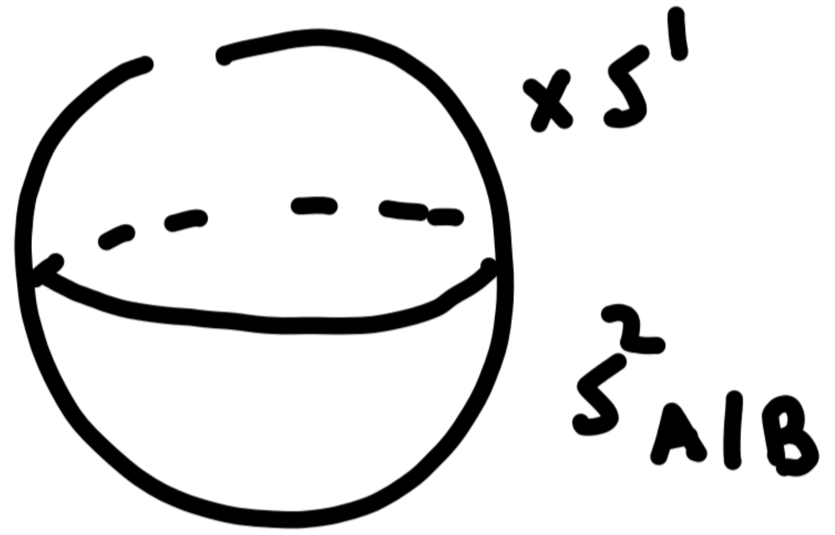
- Inner product tells us

$$\mathcal{Z}_{I\times T^2}(\mathrm{ED}_\alpha, \mathrm{ED}'_\beta) = \langle \mathcal{L}_\alpha | \mathcal{L}'_\beta \rangle = \delta_{\alpha\beta}$$

$$\mathcal{Z}_{I\times T^2}(\mathrm{EN}_\alpha, \mathrm{EN}'_\beta) = \langle \mathrm{Stab}_\alpha | \mathrm{Stab}'_\beta \rangle = \delta_{\alpha\beta}$$

- ESE is an orthornormal change of basis on $E_{\mathsf{T}_H}(\mathscr{M}_H)$

Twisted indices



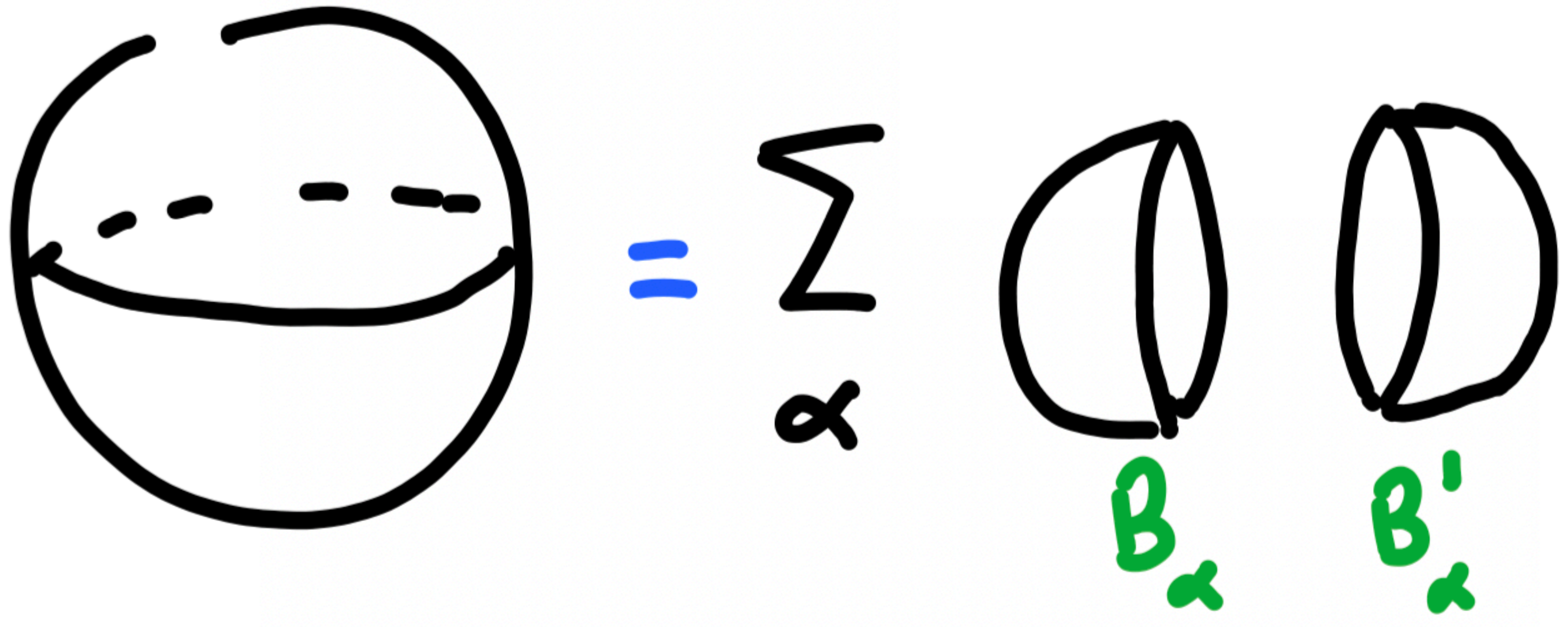
- A and B twisted indices are partition functions on $S^2 \times_{A/B} S^1$. (3d lift of A and B model)

- Can be expressed as Witten indices: $\mathcal{I}_{A,B}(\mathcal{T}) = \mathrm{Tr}_{\mathcal{H}_{S^2_{A,B}}} (-1)^F q^J t^R$

- Count states in Q_H (or Q_C) cohomology. Expect to compute sheaf cohomology groups of *unmarked (twisted)* quasimap moduli spaces.

- We can compute either index by a technique called *Coulomb branch localisation.*

- **General principle:** the path integral can be *sliced* and three manifold pfns factorise (holomorphic factorisation [Beem et. al]):



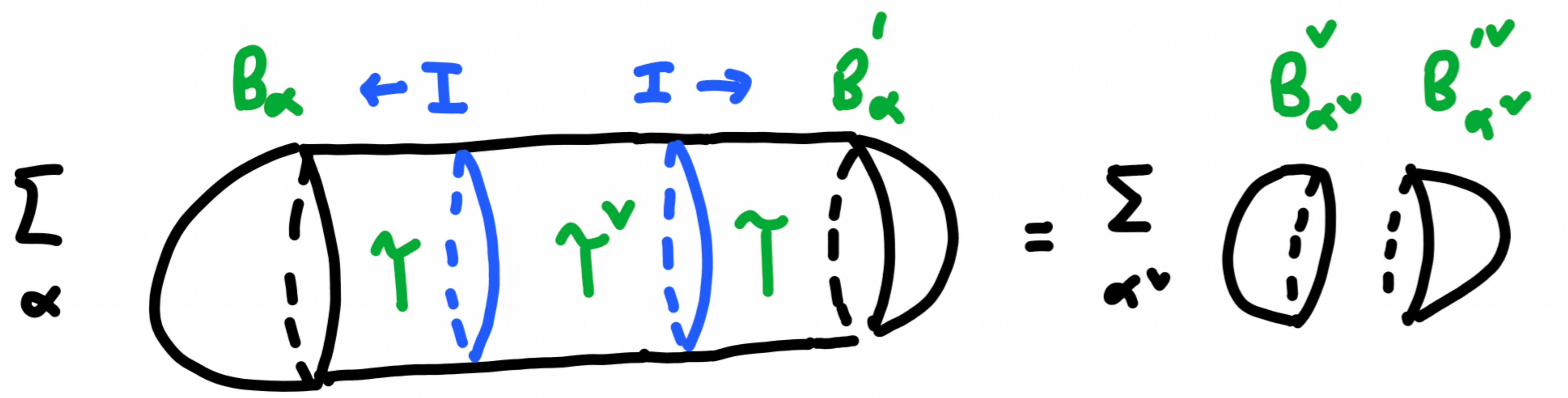
$$\mathcal{I}_A = \sum_\alpha Z_{S^1\times H^2}(B_\alpha ; t\to tq)\, Z_{S^1\times H^2}(B'_\alpha ; t\to tq)$$

$$\sim \sum_\alpha V_\alpha^{tw}\, V_\alpha^{tw\,'}$$

$$\mathcal{I}_B = \sum_\alpha V_\alpha V'_\alpha$$

Mirror symmetry of twisted indices

- Instead, expand in *stable* basis (enriched Neumann boundary conditions) and use mirror symmetry. Proof by picture:

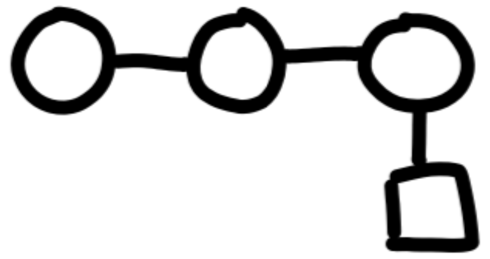


$$\mathcal{I}_B(\gamma) = \sum_\alpha V_\alpha V'_\alpha = \sum_{\alpha^\vee} \mathcal{I}(EN_{\alpha^\vee}; \gamma^\vee)\, \mathcal{I}(EN'_{\alpha^\vee}; \gamma^\vee)$$

$$= \sum \; \mathrm{stab}_{\alpha^\vee \beta^\vee} V_{\beta'}(\gamma^\vee)\, \mathrm{stab}'_{\alpha^\vee \gamma^\vee} V'_{\gamma^\vee}(\gamma^\vee)$$

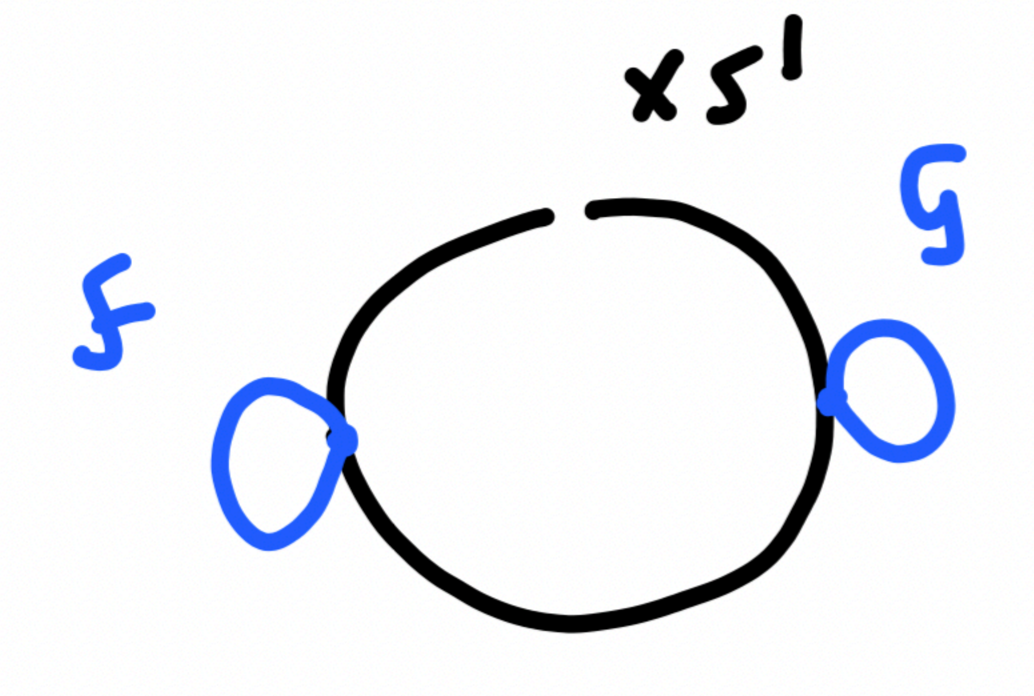
$$= \sum_{\alpha^\vee} V_{\alpha^\vee}(\gamma^\vee)\, V'_{\alpha^\vee}(\gamma^\vee) = \mathcal{I}_A(\gamma^\vee)$$

Quantum K-theory (B-twist)



$\mathfrak{g}$

- Enhance the setup with chiral operator insertions (line operators here)



$\mathcal{F}, \mathcal{G} \in K_{\mathsf{T}_H}(\mathcal{M}_H)$

physics

$$\langle \mathcal{F}\mathcal{G} \rangle = \sum_{\alpha} \mathsf{V}_{\alpha}^{(\mathcal{F})} \mathsf{V}_{\alpha}^{(\mathcal{G})}$$

- Gauge/bethe etc. [Nekrasov and Shatashvili, Okounkov et. al.] tell us: $U_{\hbar}(\hat{\mathfrak{g}}) \curvearrowright K_{\mathsf{T}_H}(\mathcal{M}_H)_{\text{Loc.}}$

- Quantum K-theory construction of [Smirnov et. al.] define a 'curve corrected' version of $K_{\mathsf{T}_H}(\mathcal{M}_H)$ with algebra structure:

$$QK_{\mathsf{T}_H}(\mathcal{M}_H) = \mathbb{Z}[s^{\pm 1}, x^{\pm 1}, t^{\pm 1}, \zeta^{\pm 1}]/\text{Bethe equations}$$

$$\mathcal{F} \circledast \mathcal{G} = \mathcal{F} \otimes \mathcal{G} + \gamma \left(\begin{matrix} \text{curve} \\ \text{corrections} \end{matrix} \right)$$

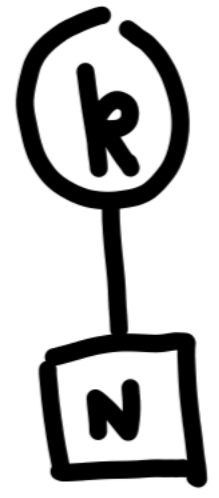
Physics: $\langle \mathcal{F} \rangle_B = \sum_{\text{Bethe}} \mathcal{F} \big|_{\alpha^*}$

- $QK_{\mathsf{T}_H}(\mathcal{M}_H)$ is generated by quantum tautological classes: $\hat{\mathcal{V}} = \mathcal{V} +$ curve corrections

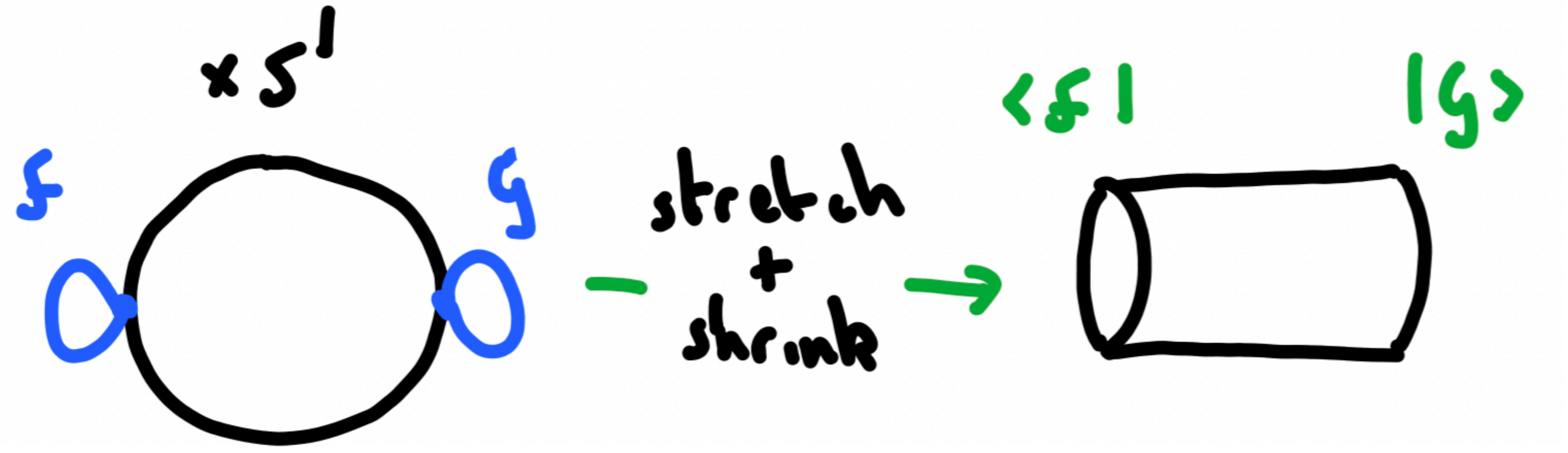
- Sometimes it happens that $\hat{\mathscr{V}} = \mathscr{V}$

Large framing vanishing $\leftrightarrow$ good/ugly

$\lim_{q\to 0,\infty} V_\alpha^{(f)}$ exists $\leftrightarrow$ monopole unitarity bound $\Delta > 1$



- In that case:



- From the point of view of holomorphic factorisation:

$$\langle f g \rangle = \sum_\alpha V_\alpha^{(f)} V_\alpha^{(g)} = \sum_\alpha f_\alpha g_\alpha \, PE[T\mathcal{M}_H] + O(\epsilon)$$

$$= \chi(f \otimes g ; \mathcal{M}_H) + O(\epsilon)$$

- No insertions then B-twisted index = Higgs branch Hilbert series!

Mirror symmetry of quantum K-theory? (A-twist?)

- Mirror symmetry of twisted indices:

  Already quite involved. Required duality interfaces and elliptic stable envelopes.

$$\begin{array}{ccc} \mathcal{I}_A(\mathcal{T}) & & \mathcal{I}_A(\mathcal{T}^\vee) \\ & \times & \\ \mathcal{I}_B(\mathcal{T}) & & \mathcal{I}_B(\mathcal{T}^\vee) \end{array} \quad \checkmark$$

- Enhancing with operator insertions, the B side is the quantum K-theory.

$$\begin{array}{ccc} \mathcal{T} & & \mathcal{T}^\vee \\ ? & & ? \\ QK_T(\mathcal{M}_H(\mathcal{T})) & \times & QK_T(\mathcal{M}_H(\mathcal{T}^\vee)) \end{array}$$

c.f. 2d homological mirror symmetry

$$\mathrm{Fuk}(X) \cong D^b_a(\hat{X})$$

- Geometrically, we've shown the $O(1)$ version of this duality is the equivalence of ordinary quasimaps of $\mathscr{M}_H$ to twisted quasimaps of $\mathscr{M}_C$.

Summary

- Demonstrated that exceptional Dirichlet boundary conditions $\mathscr{B}_\alpha$ reproduce enumerative invariants of symplectic resolutions (Operator count coincides with geometric localisation formulae).

- In particular, half indices yield Verma characters of quantised coordinate rings.

- We discussed the role of elliptic cohomology in boundary conditions of 3d $\mathscr{N} = 4$ theories. Duality interfaces can be used to derive mirror dual boundary conditions: The mirror dual of exceptional Dirichlet is enriched Neumann. (Fixed point vs. Stable basis)

- Holomorphic factorisation in physics provides novel formulae for quasimap invariants.

- The unitarity of the elliptic stable envelope implies mirror symmetry of twisted indices. It would be interesting to investigate the algebraic structure in more detail. In particular a mathematical definition of the A side is lacking.

Thanks