

Factorisation and Vortices in 3d $\mathcal{N} = 4$ Gauge Theories

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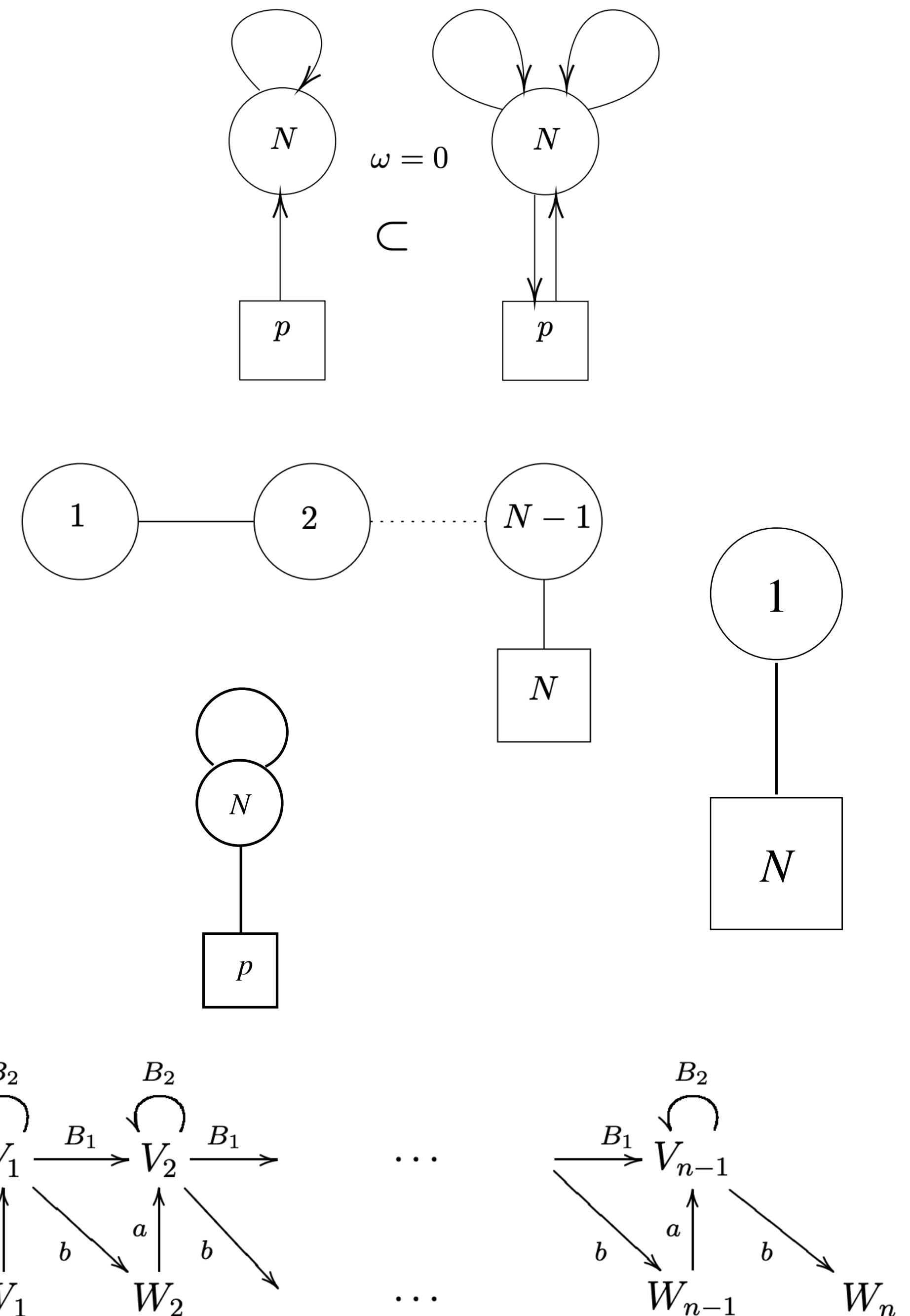
Based on 2002.04573, 2010.09732 and 2010.09741 with M. Bullimore, N. Dorey and D. Zhang

Background

- Extended algebras acting on BPS states of supersymmetric field theories in various dimensions
 - Supersymmetric indices/partition functions as characters
 - Quantum algebras acting on homology, K-theory, elliptic cohomology of quiver varieties
 - 3d mirror symmetry and symplectic duality
- Exponential $N^{3/2}$ growth of states counted by indices. AdS₄ holography — saddle points

Outline

- Quick review of 3d $\mathcal{N} = 4$ gauge theory
- Quantised Higgs and Coulomb branch algebras. Modules induced by boundary conditions.
 - Verma modules and exceptional Dirichlet
- Hemisphere partition functions $S^1 \times H^2$, factorisation and “IR formulae”
 - Concrete examples
- Twisted indices, Hilbert series and Poincaré polynomials
- 3d ADHM theory



Background on 3d $\mathcal{N} = 4$ theories

- 8 supercharges $Q_{\alpha}^{a\dot{a}}$
- Gauge group G and representation $\mathcal{R} = R \oplus R^*$
- R-symmetry $SU(2)_H \times SU(2)_C$
- Global symmetry $G_H \times G_C$
- Generic mass and FI deformations
 - $\vec{m} \in (\mathfrak{t}_H)^3$ and $\vec{t} \in (\mathfrak{t}_C)^3$
- Deformation t . Background vev for anti-diagonal R-symmetry combination $\mathcal{N} = 2^*$

3d $\mathcal{N} = 4$ Quiver Lagrangians

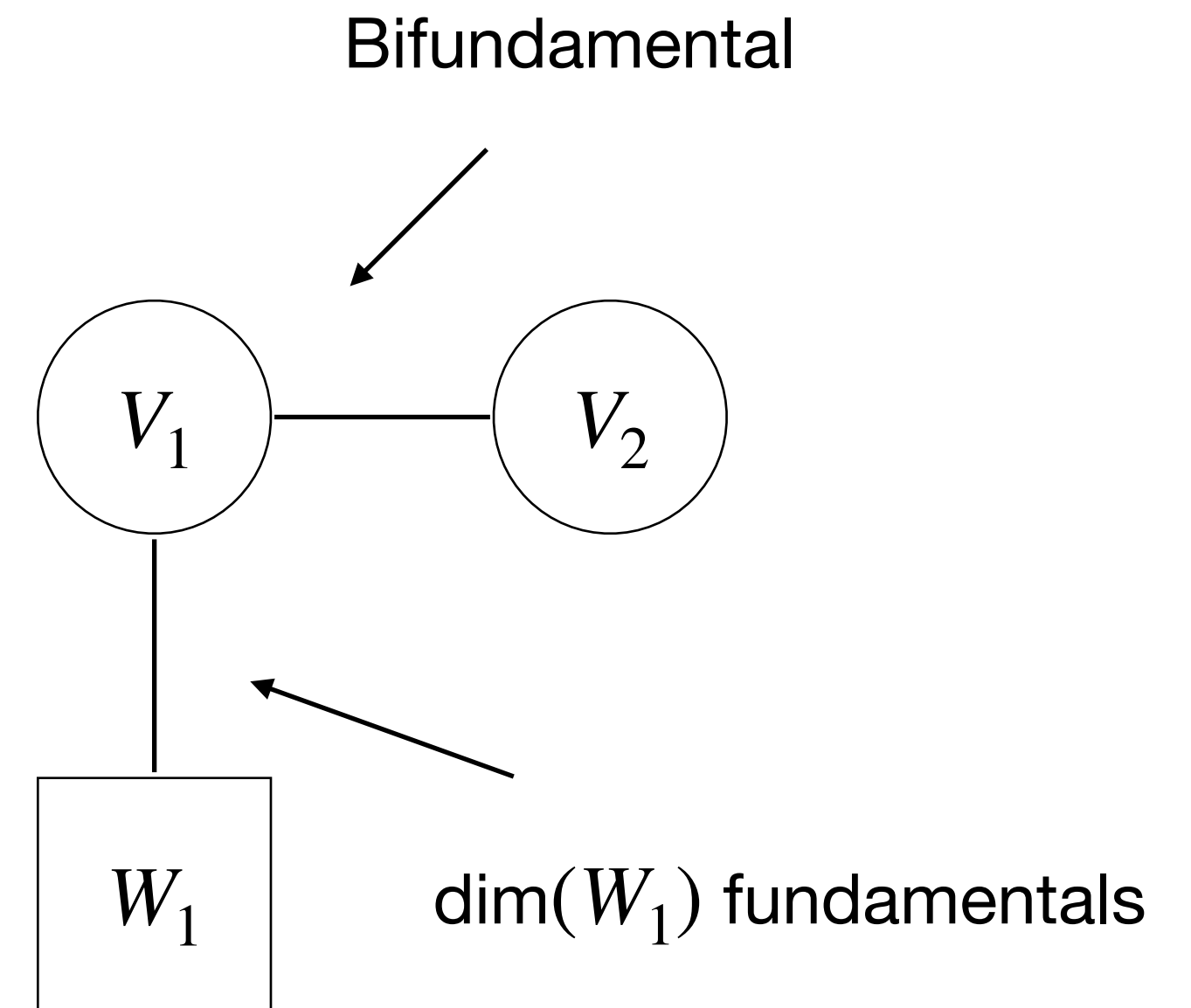
- 3d $\mathcal{N} = 4$ vectormultiplet

- A_μ, σ, φ

- 3d $\mathcal{N} = 4$ hypermultiplets

- (X, Y)

$$\mathcal{R} = T^*M$$



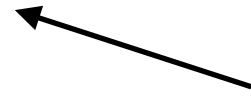
Moduli spaces of vacua

Classical



- Higgs branch $\mathcal{M}_H$ and Coulomb branch $\mathcal{M}_C$
- $\mathcal{M}_H$ and $\mathcal{M}_C$ hyperkähler
- G_H, G_C are tri-Hamiltonian isometries
- Assumption: flow to superconformal fixed point. Isolated massive vacua α
- $\mathcal{M}_H$ and $\mathcal{M}_C$ are symplectic resolutions with isolated singularities
- $m \in \mathfrak{t}_H$ and $\xi \in \mathfrak{t}_C$ are resolution and deformation parameters.

Vector multiplet scalars with monopoles $V_{\pm}$



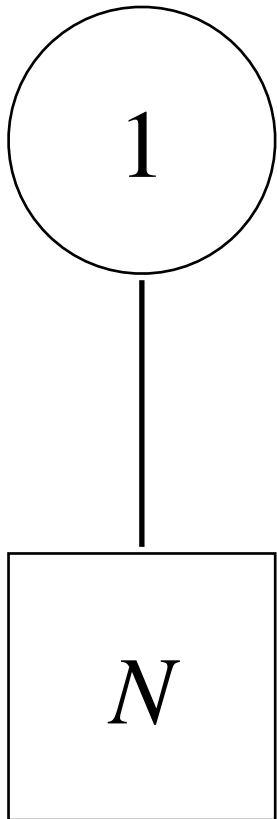
$$\mu_{\mathbb{R}} = X \cdot X^{\dagger} - Y \cdot Y^{\dagger}$$

$$\mu_{\mathbb{C}} = X \cdot Y$$

$$\mathcal{M}_H = \{\mu_{\mathbb{C}} = 0, \mu_{\mathbb{R}} = \xi\} / G$$

Examples

SQED[N]



$$\mathcal{M}_H = T^*\mathbb{P}^N$$

$$\mu_{\mathbb{R}} = \sum |X_i|^2 - |Y_i|^2$$

$$\mu_{\mathbb{C}} = \overrightarrow{X} \cdot \overrightarrow{Y}$$

$U(1)$ gauge theory.

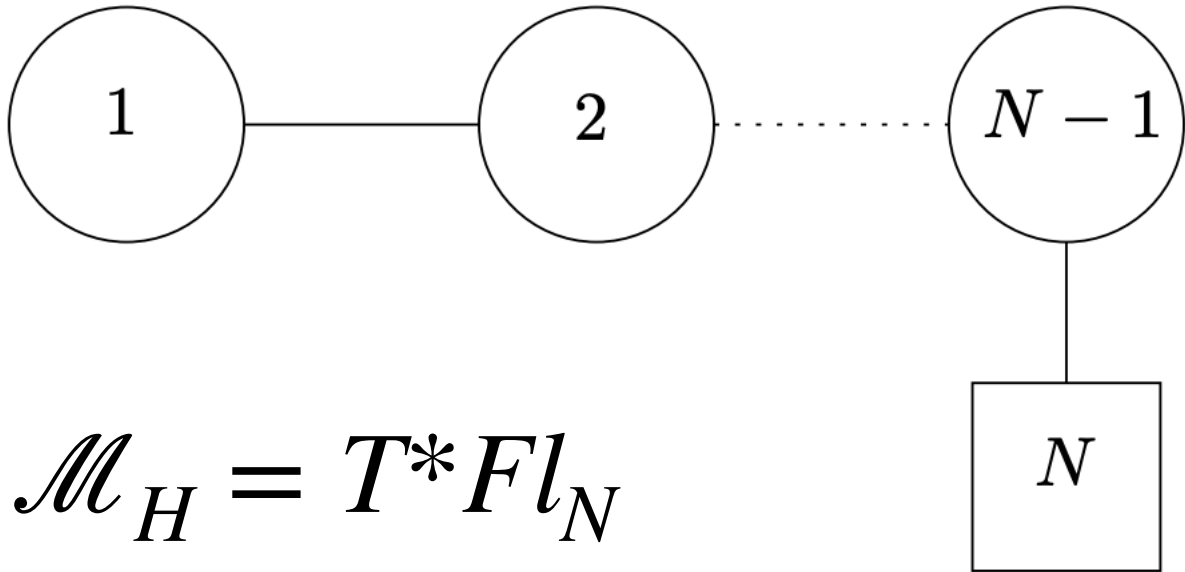
Fundamental hypermultiplets (X_i, Y_i) with $i = 1, \dots, N$

$$G_H = SU(N) \text{ and } G_C = U(1)$$

Masses $m_1, \dots, m_N$ and FI parameter η

N isolated vacua

$T[SU(N)]$



$$\mathcal{M}_H = T^*Fl_N$$

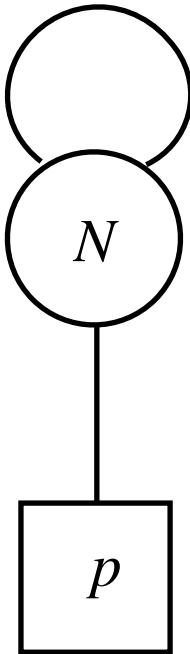
$$G = U(1) \times \dots \times U(N-1)$$

$$G_H = G_C = SU(N)$$

Masses $m_1, \dots, m_N$ and FI parameters $\eta_1, \dots, \eta_N$

Vacua labelled by $\sigma \in S_N$

$$x = e^m \text{ and } \zeta = e^\eta$$



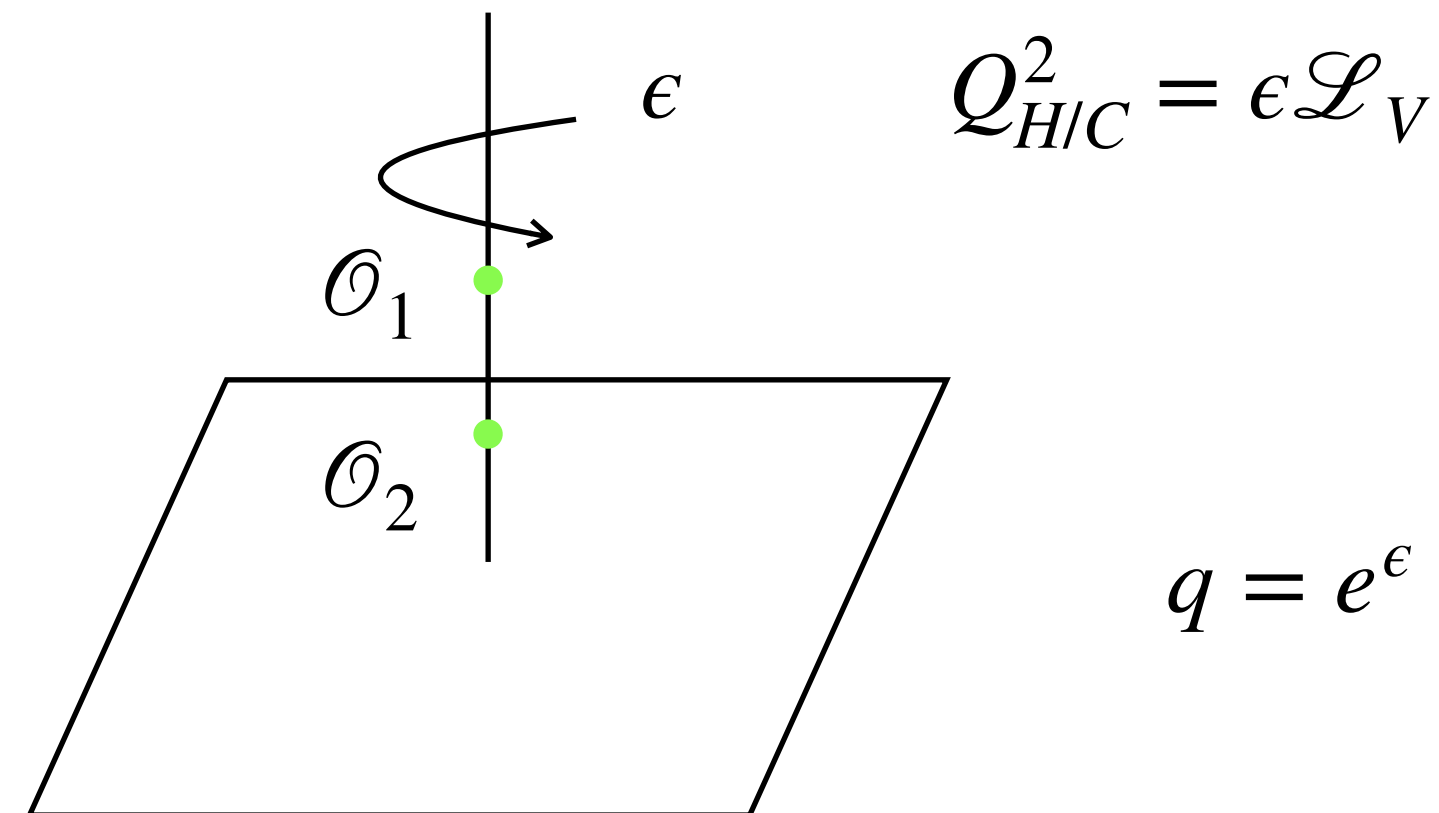
Higgs and Coulomb algebras

- Fix $\mathcal{N} = 2$ subalgebra $U(1)_H \times U(1)_C \subset SU(2)_H \times SU(2)_C$
- Ring of chiral operators/holomorphic functions $\mathbb{C}[\mathcal{M}_H]$ and $\mathbb{C}[\mathcal{M}_C]$
- Hyperkähler geometry equips with Poisson bracket

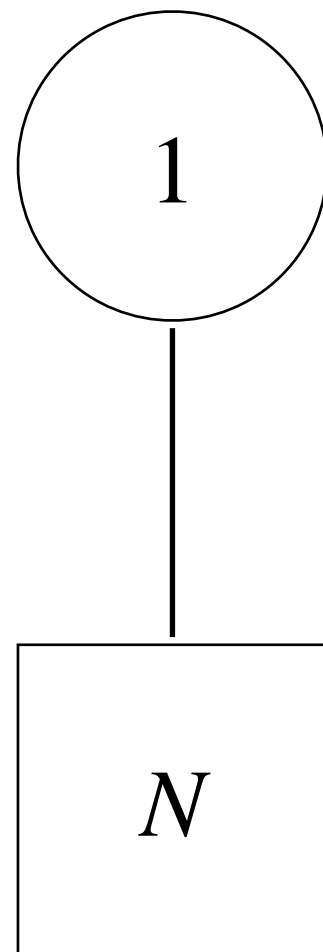
Quantisation

Ω background quantises chiral rings

$$\hat{\mathbb{C}}[\mathcal{M}_H] \text{ and } \hat{\mathbb{C}}[\mathcal{M}_C]$$



Example: SQED[N]



Higgs algebra $\hat{\mathbb{C}}[\mathcal{M}_H]$

Generated by X_i and Y_i

P.B. quantised: $[\hat{Y}_i, \hat{X}_j] = \epsilon \delta_{ij}$

Moment map $\sum_{i=1}^N : \hat{X}_i \hat{Y}_i := t_{\mathbb{C}}$

Central quotient of $U(\mathfrak{sl}_N)$

$$e_{ij} = \hat{X}_i \hat{Y}_j, \quad i < j$$

$$f_{ij} = \hat{X}_i \hat{Y}_j, \quad i > j$$

$$h_j = \hat{X}_j \hat{Y}_j - \hat{X}_{j+1} \hat{Y}_{j+1} \quad j = 1, \dots, N-1$$

Coulomb algebra $\hat{\mathbb{C}}[\mathcal{M}_C]$

Generated by complex scalar φ and monopole operators $v_{\pm}$

$$[\hat{\varphi}, \hat{v}_{\pm}] = \pm \epsilon \hat{v}_{\pm}$$

$$\hat{v}_+ \hat{v}_- = \prod_{i=1}^N (\hat{\varphi} + m_{i,\mathbb{C}} - \frac{\epsilon}{2})$$

$$\hat{v}_- \hat{v}_+ = \prod_{i=1}^N (\hat{\varphi} + m_{i,\mathbb{C}} + \frac{\epsilon}{2})$$

Spherical rational Cherednik algebra — finite W algebra

We will discuss Verma modules later!

Vortex moduli spaces

- Theory admits $\frac{1}{2}$ BPS vortex solutions
- Hilbert space of Ω -deformed theory in plane with mass deformations = Equivariant homology of VMS.
[Bullimore, Dimofte, Gaiotto, Hilburn, Kim]
- Relation between vortices and quasi-maps.
- Kähler manifolds with isometries x, q

Vortices are labelled by $\mathbf{d} = \frac{1}{2\pi} \int_{S^2} \text{tr} F_i$ $X, Y \rightarrow \mathcal{M}_H$ at infinity

Identify $\mathbf{d} \in H_2(\mathcal{M}_H, \mathbb{Z})$

← Quasimap degree

Algebraic description



$\text{QM}_\alpha^d(\mathbb{P}^1 \rightarrow \mathcal{M}_H)$ [Okounkov]

K-theoretic vertex functions

q rotates $\mathbb{P}^1$

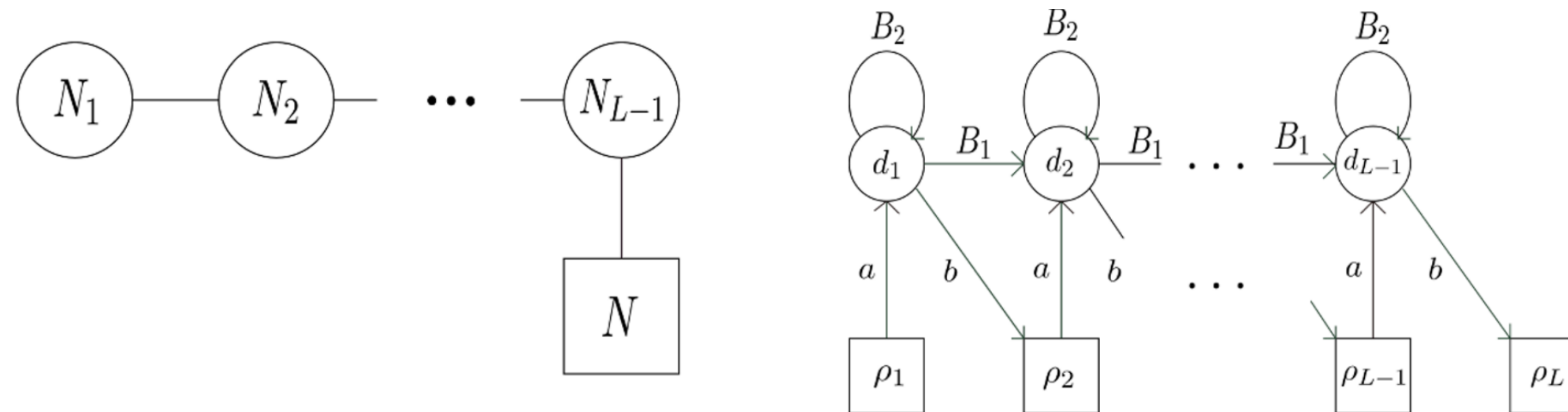
G_H global symmetries

Isolated fixed points

Example: Laumon space/Handsaw quiver

Laumon spaces resolution of singularities $\mathfrak{Q}_\alpha^d = \mathrm{QM}_\alpha^d(\mathbb{P}^1 \rightarrow \text{flag})$

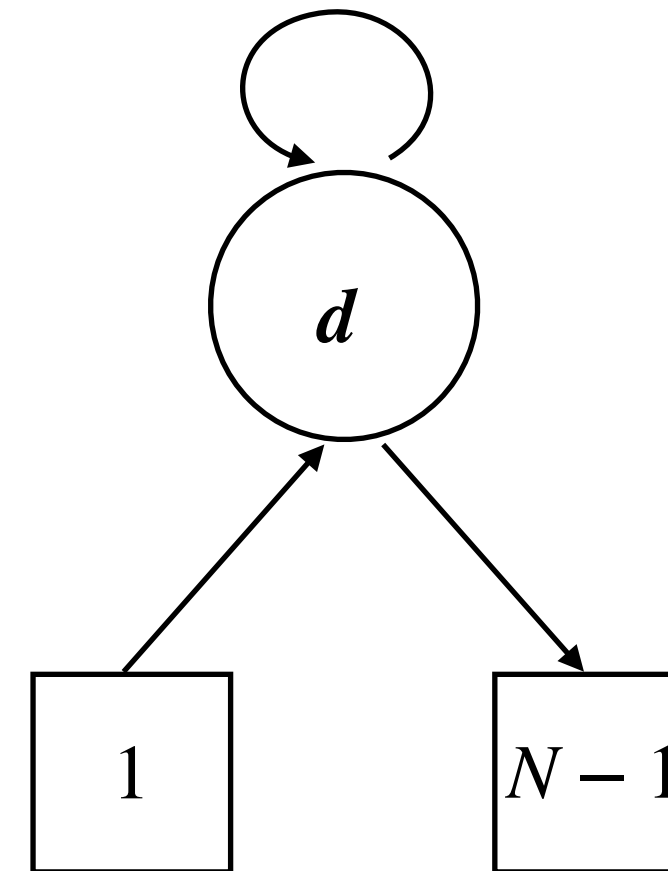
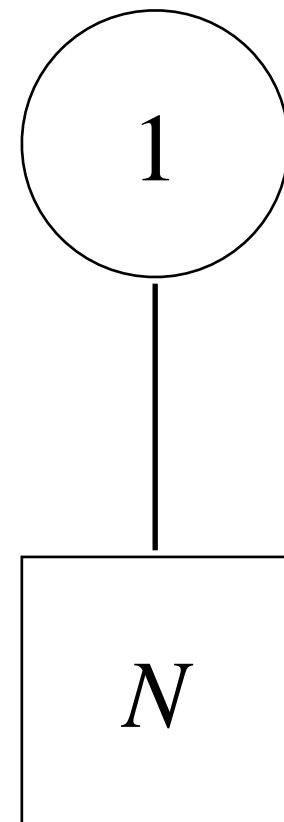
Realisation as handsaw quiver variety [Nakajima]



Will discuss χ_t genera and Poincaré polynomials

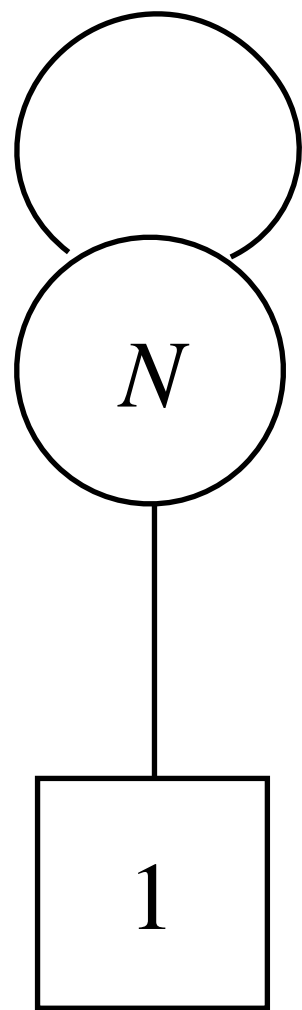
Example

- SQED[N] vortex moduli space



$\mathbb{C}^{dN}$

Another Example



$$\mathcal{M}_H = \text{Hilb}^N(\mathbb{C}^2)$$

- $\text{QM}_\lambda^d(\mathbb{P}^1 \rightarrow \text{Hilb}^N(\mathbb{C}^2))$
- Smooth quiver description?
- $\chi(\hat{\mathcal{O}}_{\text{Vir}})$ — Localisation formula

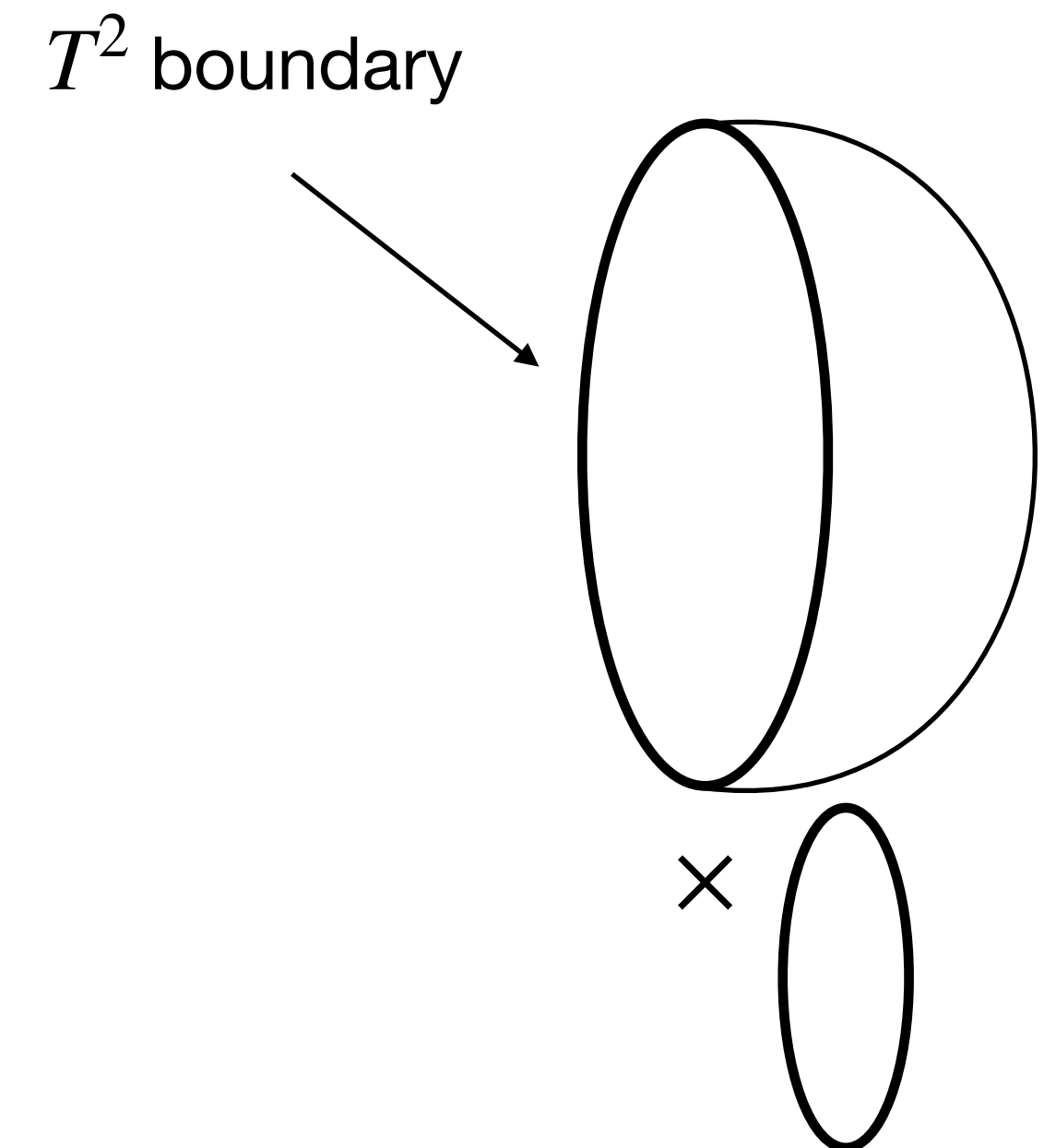
Hemisphere partition function

- We compute partition function on hemisphere $S^1 \times H^2$
- $\mathcal{N} = (2,2)$ boundary condition $\mathcal{B}$ on T^2

$$\mathcal{Z}_{S^1 \times H^2} = \mathcal{Z}_{\text{Classical}} \mathcal{Z}_{\text{1-loop}} \mathcal{Z}_{\text{Vortex}} \longleftarrow \mathcal{Z}_{\text{Vortex}} \sim \chi(\hat{\mathcal{O}}_{\text{Vir.}})$$

[Benini and Peelaers] [Fujitsuka, Honda and Yoshida]

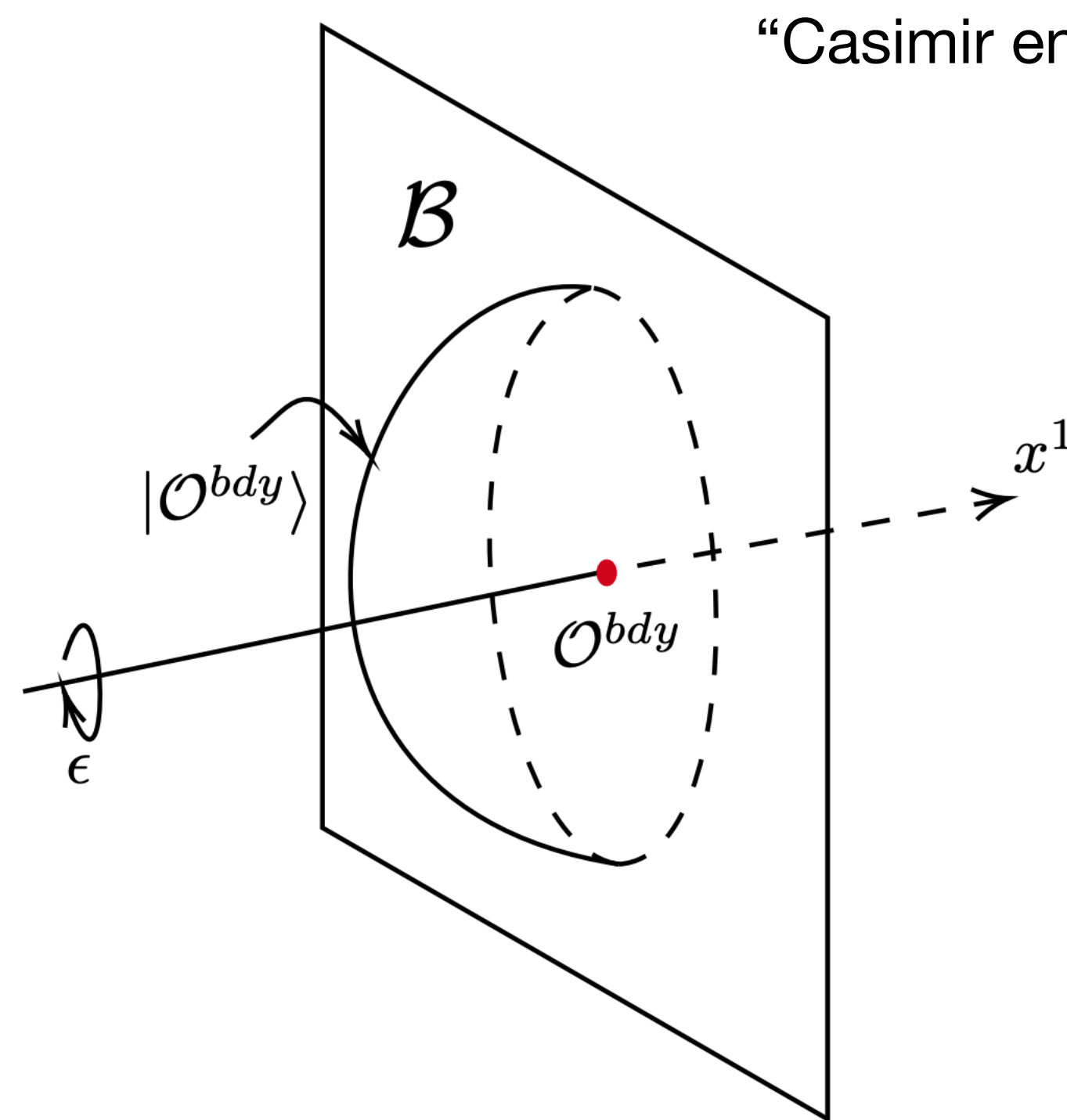
Modification of [Yoshida and Sugiyama]—with particular B.C.



$$ds^2 = d\tau^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

Twisted boundary conditions $\tau \sim \tau + \beta r$

State-operator map



“Casimir energy”

- Relate the hemisphere partition function to count of local operators.

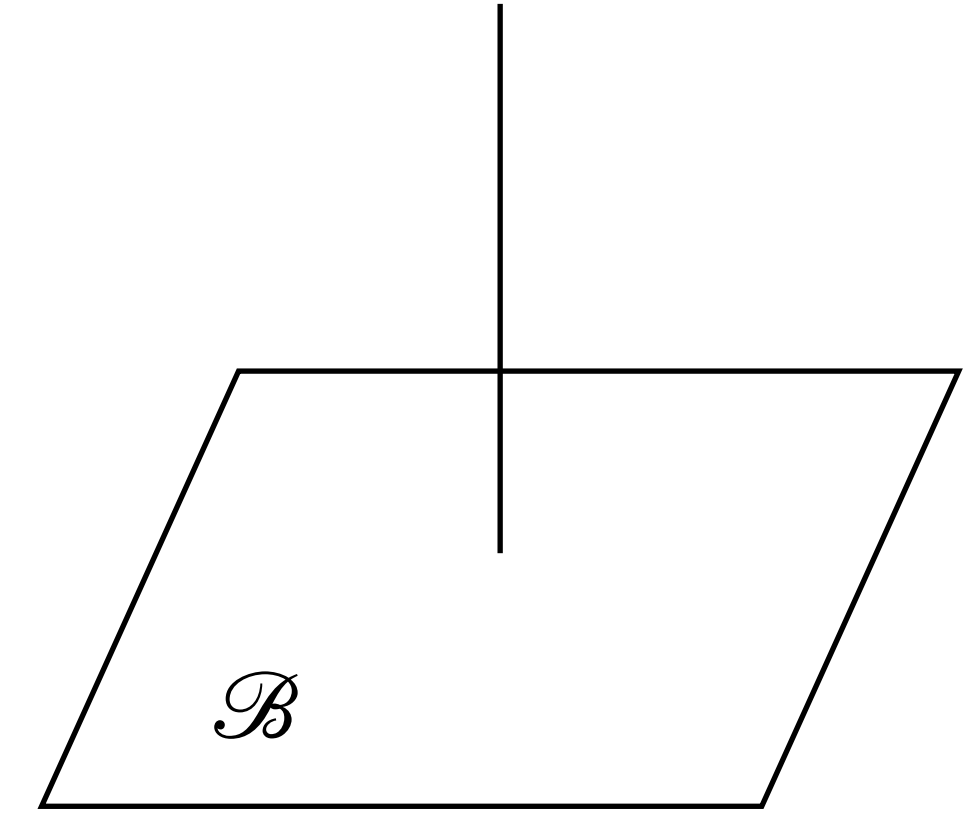
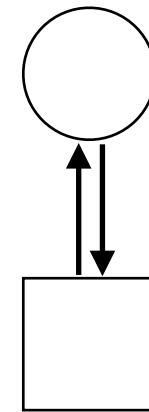
$$\mathcal{Z}_{S^1 \times H^2} = e^{\phi} \mathcal{I}$$

Half index

- $\text{Tr}_{\mathcal{H}_{\mathcal{B}}} (-1)^F q^{J + \frac{R_V + R_A}{2}} t^{\frac{R_V - R_A}{2}} x^{F_H} \xi^{F_C}$
- We will compute half index

Exceptional Dirichlet

- Boundary conditions $\mathcal{B}_\alpha$ associated to each vacua α
- Dirichlet $\mathcal{D}$ for $\mathcal{N} = 4$ vector multiplet
- Lagrangian splitting of the hypers $L \oplus L^*$
- $Y_L|_\partial = c_L$
- Fully breaks G , preserves T_H and T_C
- L chosen to give $\mathcal{L}_\alpha \subset \mathcal{M}_H$



Vector multiplet

Recipe for computing half index
[Dimofte, Gaiotto and Paquette]

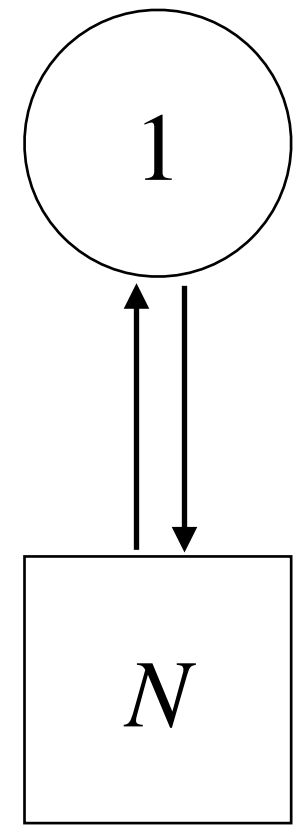
$$\frac{(q^{\frac{1}{2}}t^2; q)_\infty}{(q)_\infty} \sum_{m \in \mathbb{Z}} y^{k_{\text{eff}} m} \times [\text{matter index}](q^m u)$$

Boundary monopoles

Boundary hypers

Example: SQED[N]

Fix chamber $\mathfrak{C}_H = \{m_1 < \dots < m_N\}$ and $\mathfrak{C}_C = \{\xi > 0\}$



$$\begin{aligned} \partial_{\perp} Y_j &= 0, & X_j &= c\delta_{ij} & j &\leq i \\ \partial_{\perp} X_j &= 0, & Y_j &= 0 & j &> i \end{aligned}$$

i specifies particular vacuum

First compute Dirichlet half-index:

$$\sum_{m \in \mathbb{Z}} \left(\xi \left(q^{\frac{1}{4}} t^{-\frac{1}{2}} \right)^{2i-N} \right)^m \frac{(tq^{\frac{1}{2}}; q)_{\infty}}{(q; q)_{\infty}} \prod_{j \leq i} \frac{\left(q^{\frac{3}{4}+m} t^{-\frac{1}{2}} z^{-1} x_j^{-1}; q \right)_{\infty}}{\left(q^{\frac{1}{4}+m} t^{\frac{1}{2}} z^{-1} x_j^{-1}; q \right)_{\infty}} \prod_{j > i} \frac{\left(q^{\frac{3}{4}-m} t^{-\frac{1}{2}} z x_j; q \right)_{\infty}}{\left(q^{\frac{1}{4}-m} t^{\frac{1}{2}} z x_j; q \right)_{\infty}}.$$

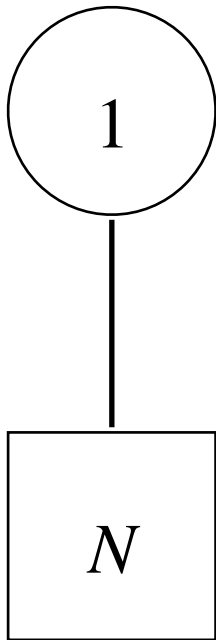
Effective CS level

Boundary symmetry $U(1)_{\partial}$

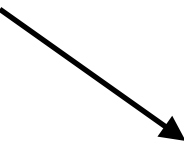
Expansions in q spin

Specialise fugacity $z = x_i^{-1} t^{-\frac{1}{2}} q^{-\frac{1}{4}}$ for non-zero chiral breaking combination of flavour, gauge and R-symmetry

Result: SQED[N]

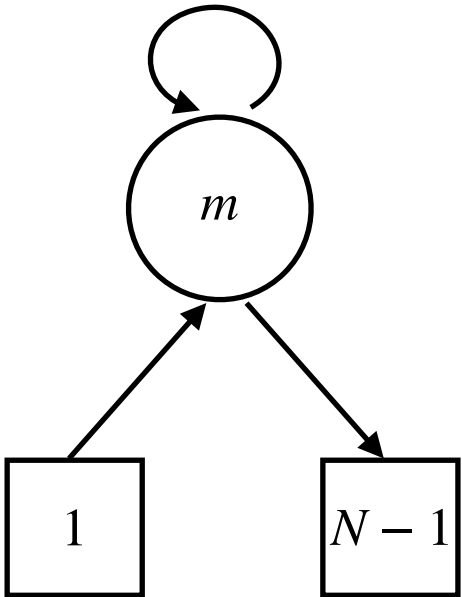


From the hemisphere localisation



$$\mathcal{Z}_{\mathcal{B}_i} = \mathcal{Z}_i^{\text{Cl}} \mathcal{Z}_i^{\text{1-loop}} \mathcal{Z}_i^{\text{Vortex}}$$

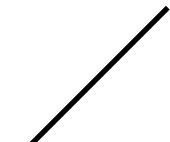
- Vortex moduli space is handsaw quiver



$$\mathbb{C}^{mN}$$

1 fixed point

Different



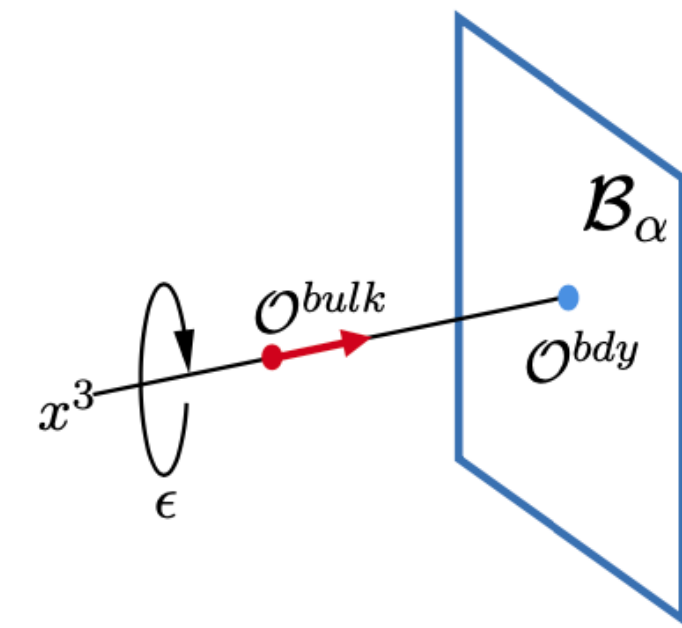
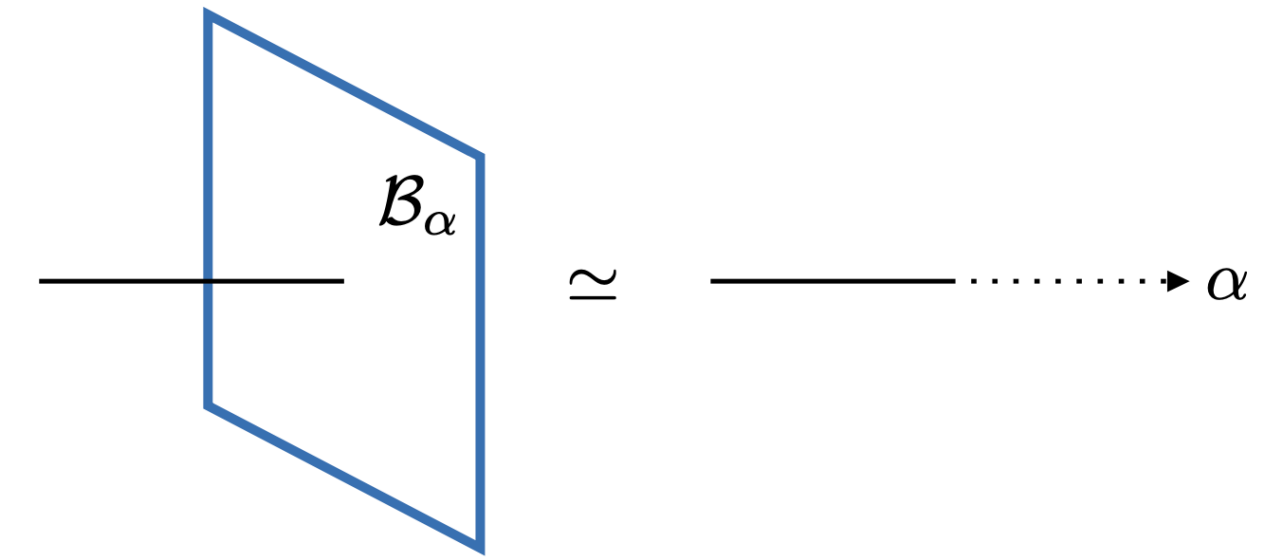
$$\mathcal{Z}_i^{\text{1-loop}} = \prod_{j=1}^{i-1} \frac{\left(q \frac{x_i}{x_j}; q\right)_{\infty}}{\left(q^{\frac{1}{2}} t \frac{x_i}{x_j}; q\right)_{\infty}} \prod_{j=i+1}^N \frac{\left(q^{\frac{1}{2}} t^{-1} \frac{x_j}{x_i}; q\right)_{\infty}}{\left(\frac{x_j}{x_i}; q\right)_{\infty}},$$

$$\mathcal{Z}_i^{\text{Vortex}} = \sum_{m \geq 0} \left(\left(q^{\frac{1}{4}} t^{-\frac{1}{2}} \right)^N \xi \right)^m \prod_{j=1}^N \frac{\left(q^{\frac{1}{2}} t \frac{x_i}{x_j}; q \right)_m}{\left(q \frac{x_i}{x_j}; q \right)_m}.$$

$$\mathcal{Z}_i^{\text{Vortex}} = \sum_{d \geq 0} \zeta^d \chi_t(\mathfrak{Q}_{\alpha}^d)$$

Why exceptional Dirichlet?

- Exceptional Dirichlet mimics vacuum at infinity—factorisation
- Flow to thimbles in IR RW sigma model—associated to vacua
- General principle [Bullimore, Dimofte, Gaiotto and Hilburn] that $\mathcal{B}$ yields modules for $\hat{\mathcal{C}}[\mathcal{M}_H]$ and $\hat{\mathcal{C}}[\mathcal{M}_C]$
- Exceptional Dirichlet $\mathcal{B}_\alpha$ yields Verma modules



“Casimir” e^ϕ $\longleftrightarrow$ Highest weight of Verma

- $\mathcal{N} = 4$ limits of hemisphere partition functions are Verma module characters

Action on boundary operators $\longleftrightarrow$ Equivariant homology VMS

Specialised limits

$$\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}} (-1)^F q^{J + \frac{R_V + R_A}{4}} t^{\frac{R_V - R_A}{2}} x^{F_H} \xi^{F_C}$$

B-limit: $t \rightarrow q^{-\frac{1}{2}}$

A-limit: $t \rightarrow q^{\frac{1}{2}}$

$$\mathcal{I}_{\mathcal{B}}^{(B)} := \lim_{t^{\frac{1}{2}} \rightarrow q^{-\frac{1}{4}}} \mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}^{(B)}} x^{F_H}$$

$$\mathcal{I}_{\mathcal{B}}^{(A)} := \lim_{t^{\frac{1}{2}} \rightarrow q^{\frac{1}{4}}} \mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}^{(A)}} \xi^{F_C}$$

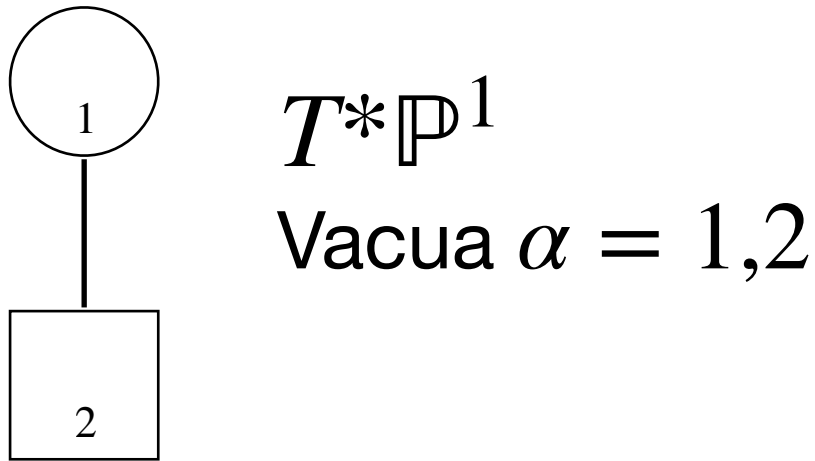
4 supercharges: Higgs operators (1-loop only)

4 supercharges: Coulomb operators (vortex only)

$e^{\phi_{\alpha}}$ $\longrightarrow$ “Equivariant vacuum energy”

Example: $T[SU(2)]$

Coulomb side

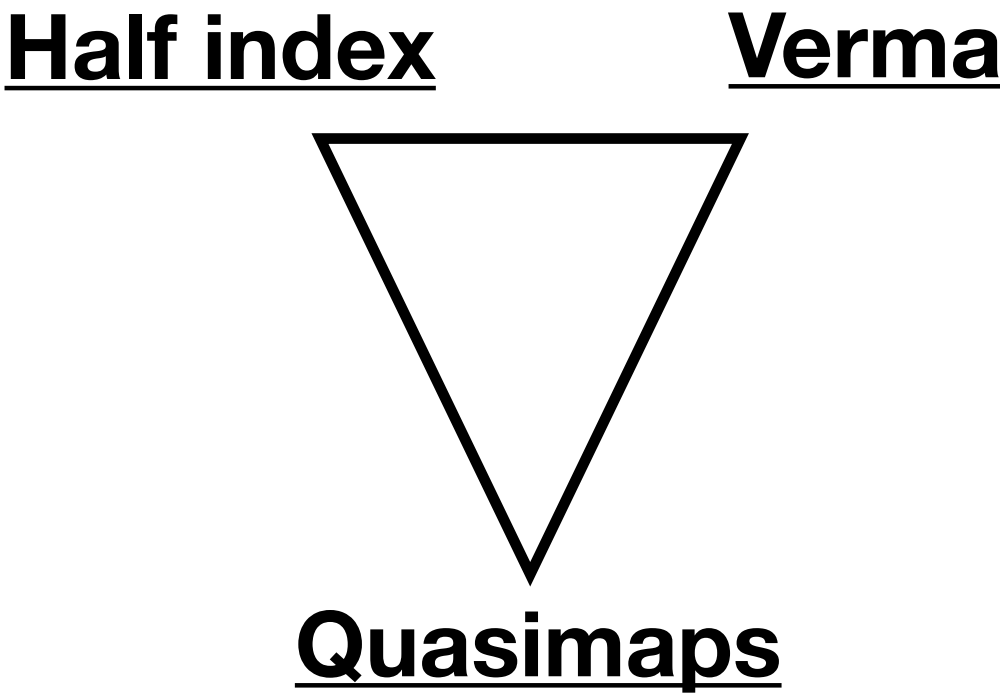


$$\mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m \geq 0} \left(q^{\frac{1}{2}} t^{-\frac{1}{4}} \xi \right)^m \prod_{j=1}^2 \frac{(q^{\frac{1}{2}} t x_\alpha / x_j; q)_m}{(q x_\alpha / x_j; q)_m}$$

A-limit

$$\chi_\alpha^C = e^{\frac{\log(\xi)\log(x_\alpha)}{\log(q)} + \frac{1}{2}} \frac{1}{1 - \xi}$$

Highest weight



$$[\hat{\phi}, \hat{v}_\pm] = \pm \epsilon \hat{v}_\pm$$

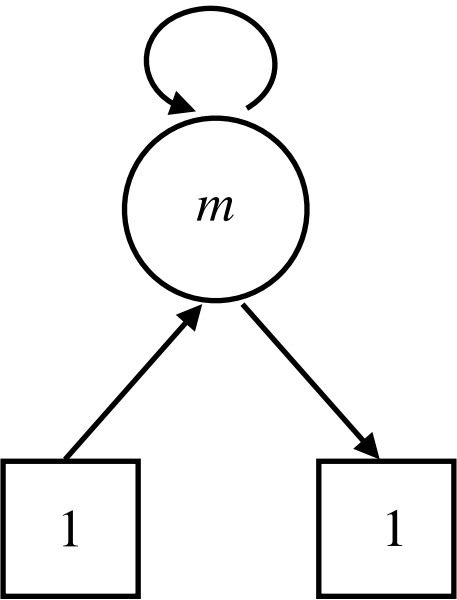
$$\hat{v}_+ \hat{v}_- = (\hat{\phi} + m_{1,\mathbb{C}} - \frac{\epsilon}{2})(\hat{\phi} + m_{2,\mathbb{C}} - \frac{\epsilon}{2})$$

$$\hat{v}_- \hat{v}_+ = (\hat{\phi} + m_{1,\mathbb{C}} + \frac{\epsilon}{2})(\hat{\phi} + m_{2,\mathbb{C}} + \frac{\epsilon}{2})$$

$$(\hat{\phi} + m_\alpha + \frac{\epsilon}{2}) | \mathcal{B}_\alpha \rangle = 0, \quad \hat{v}^+ | \mathcal{B}_\alpha \rangle = 0$$

Lower with $\hat{v}_-$

$$\mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m \geq 0} \xi^m \chi_t(\mathfrak{Q}_\alpha^m) \xrightarrow{\text{A-limit}} \mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m \geq 0} \xi^m (\# \text{ f.p.})$$



Recap so far

- Compute hemisphere partition functions and half indices with exceptional Dirichlet boundary conditions
- Specialised limits give Verma modules of $\hat{\mathcal{C}}[\mathcal{M}_H]$ and $\hat{\mathcal{C}}[\mathcal{M}_C]$

$$\lim_{t \rightarrow q^{\pm \frac{1}{2}}} \mathcal{Z}_\alpha(x, \xi; q, t) = \chi_\alpha^{\text{H,C}}(x \text{ or } \xi)$$

Factorisation

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} H_{\alpha} \tilde{H}_{\alpha}$$

- H_{α} holomorphic block [Beem, Dimofte and Pasquetti] — 3d analogue of tt^* setup

- Gluing corresponds to Heegaard decomposition of $\mathcal{M}_3$

- Demonstrated in various cases by Coulomb branch localisation

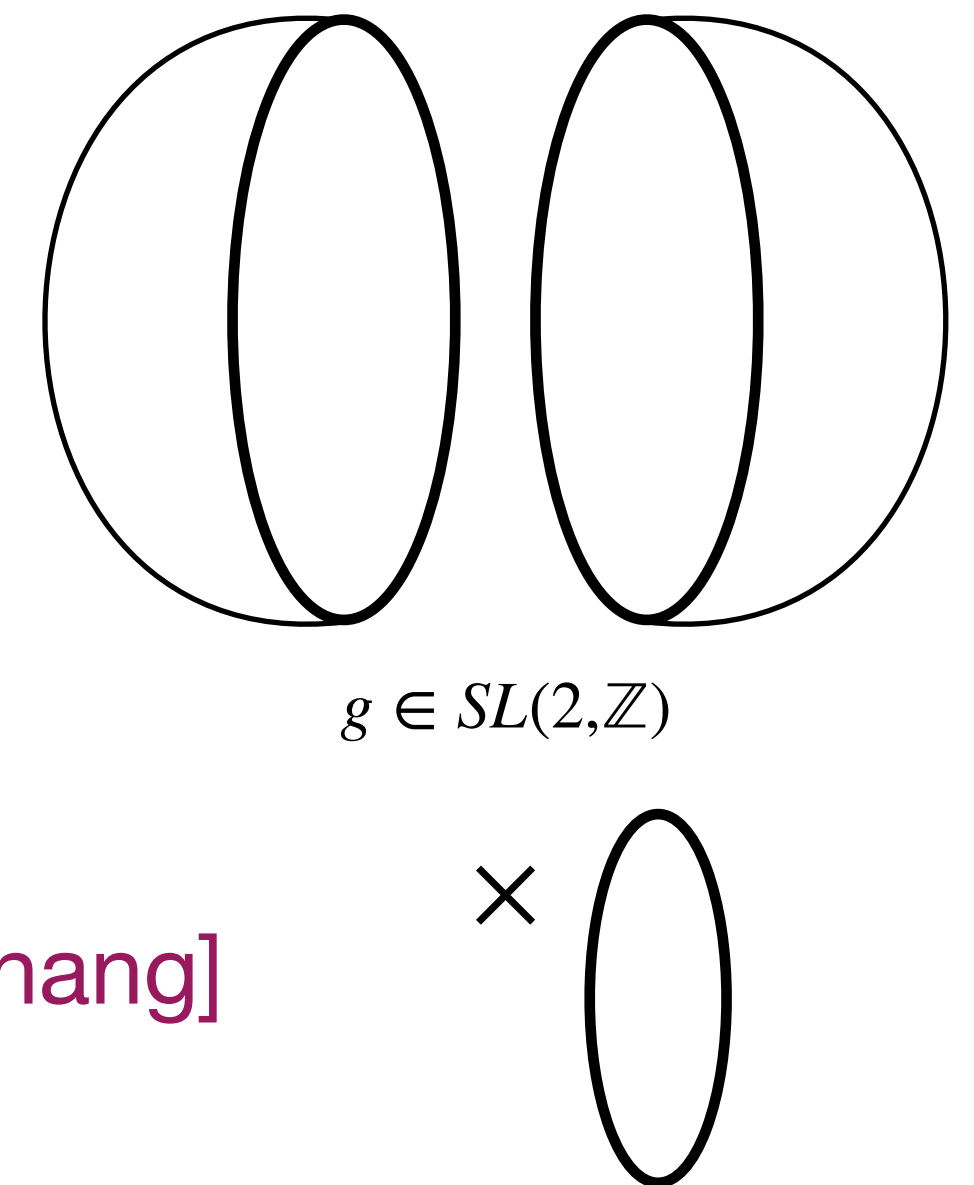
- Examples include $\mathcal{M}_3 = S^2 \times_{A,B} S^1$, S_b^3 , $S^2 \times S^1$

- Factorisation into hemisphere is *exact*

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} \mathcal{Z}_{\text{Pert.}} \mathcal{Z}_{\text{Vor.}} \mathcal{Z}_{\text{Anti vor.}}$$

Factorise
↙

- Demonstrated for SQED[N] and ADHM [SC, Bullimore, Zhang] and [SC, Dorey, Zhang]



Application: IR formulae

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} \mathcal{Z}_{S^1 \times H^2}^{\alpha} \tilde{\mathcal{Z}}_{S^1 \times H^2}^{\alpha} \qquad \lim_{t \rightarrow q^{\pm \frac{1}{2}}} \mathcal{Z}_{\alpha}(x, \xi, q, t) = \chi_{\alpha}^{H,C}(x \text{ or } \xi)$$

In the specialised $\mathcal{N} = 4$ limits we find e.g.

$$\mathcal{Z}_{\text{SC}}^B = \sum_{\alpha} \chi_{\alpha}^H(x) \chi_{\alpha}^H(x^{-1}), \qquad \mathcal{Z}_{\text{SC}}^A = \sum_{\alpha} \chi_{\alpha}^C(\xi) \chi_{\alpha}^C(\xi^{-1}),$$

$$\mathcal{Z}_{\text{tw}}^B = \sum_{\alpha} \chi_{\alpha}^H(x) \chi_{\alpha}^H(x), \qquad \mathcal{Z}_{\text{tw}}^A = \sum_{\alpha} \chi_{\alpha}^C(\xi) \chi_{\alpha}^C(\xi)$$

$$\mathcal{Z}_{S^3} = \sum_{\alpha} \hat{\chi}_{\alpha}^H(x) \hat{\chi}_{\alpha}^C(\xi),$$

← [Gaiotto and Okazaki]

A and B twisted indices

Choose $R_A = 2U(1)_H$ or $R_B = 2U(1)_C$ and place theory on $S^2 \times_{A,B} S^1$ with background R-symmetry flux

↓ E.g. Coulomb branch localisation [Benini and Zaffaroni]

$$\mathcal{Z}_{S^2 \times_{A,B} S^1} = \sum_{\alpha} H_{\alpha}(x, \xi; q, t) H_{\alpha}(x, \xi; q^{-1}, t)$$

Geometry [SC, Dorey and Zhang]

$$\sum_d \xi^d \chi_t(Q^d) = \mathcal{Z}_{S^2 \times_A S^1} = \sum_{\alpha} H_{\alpha} \tilde{H}_{\alpha} = \sum_{\alpha} \left\| \sum_d \xi^d \chi_t(\mathfrak{Q}_{\alpha}^d) \right\|^2$$

Global Laumon space

Angular momentum q is “ Q -exact”



χ_t genus of compact space

Poincaré polynomial limit

$$\mathcal{Z}_{S^1 \times H^2} = \mathcal{Z}_{\text{Classical}} \mathcal{Z}_{\text{1-loop}} \mathcal{Z}_{\text{Vortex}}$$

$$\mathcal{Z}_{\text{Vortex}}(x, \xi; q, t) \quad \begin{array}{c} q \rightarrow 0 \\ \swarrow \end{array} \quad \begin{array}{c} \searrow qt \rightarrow 0 \\ \mathcal{Z}_{\text{1-loop}}(t, x) = \text{Generating function} \end{array}$$

$$\sum_{k \geq 0} \xi^k \sum_{\text{f.p.}} t^{\text{Dim. attracting set}}$$

[Bialynicki-Birula]

$$\lim_{q \rightarrow 0} \chi_t(\mathfrak{Q}) = P_t(\pi^{-1}(0))$$

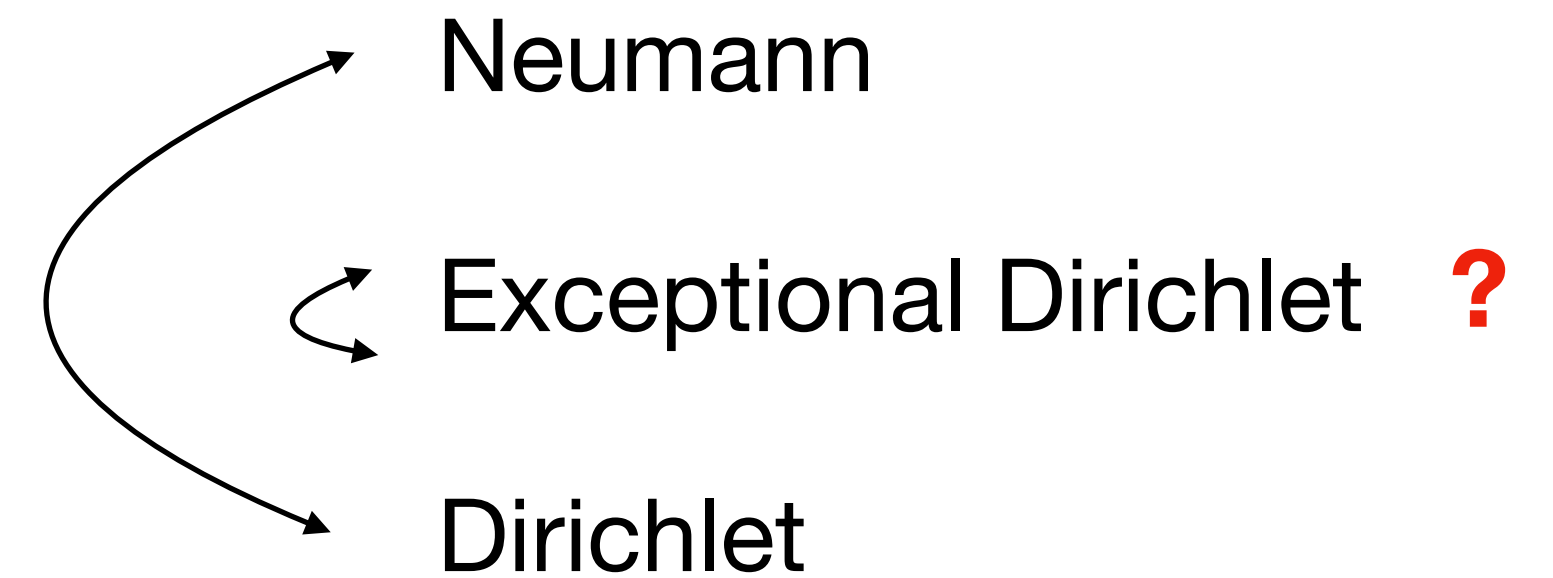
Poincaré polynomial

- Only vortices contribute in this limit — R-charge graded Verma characters
- Mirror limit only 1-loop contributions — generating function

3d mirror symmetry $X \leftrightarrow Y$

- IR duality of 3d gauge theories [Intriligator and Seiberg]

$$\mathcal{M}_H \leftrightarrow \mathcal{M}_C \quad m \leftrightarrow \xi \quad t \leftrightarrow t^{-1}$$



- Boundary conditions transform non-trivially

$$\mathcal{Z}_\alpha \rightarrow U_{\alpha\beta} \mathcal{Z}_\beta$$

Elliptic stable envelopes

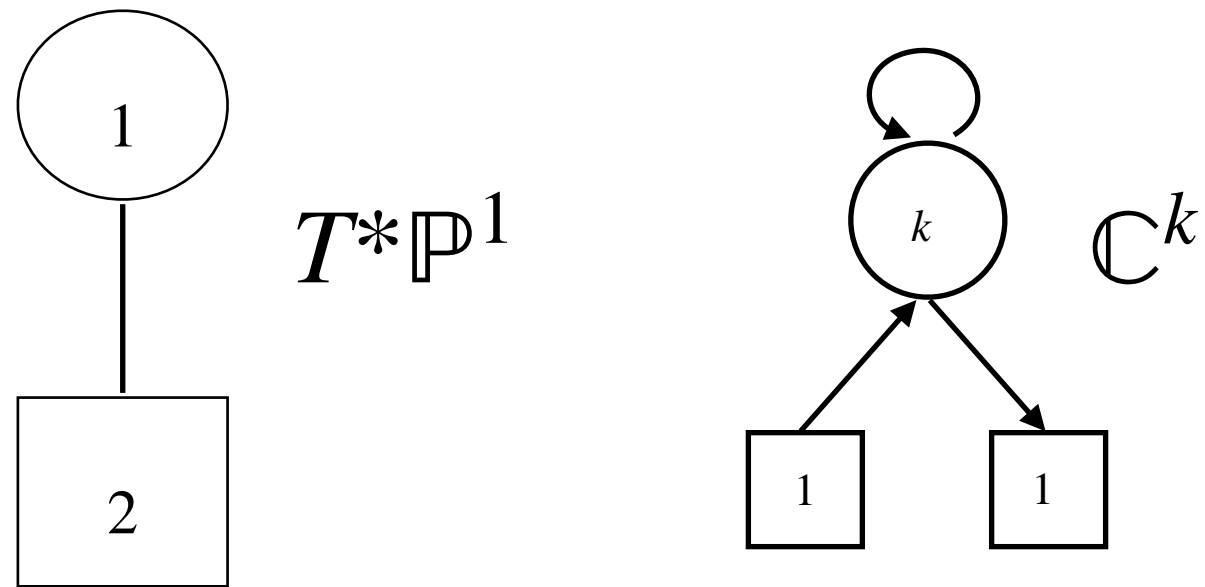
- Two simplifying limits

$$\lim_{qt^{\pm \frac{1}{2}} \rightarrow 0} \mathcal{I}_\alpha^X = \lim_{qt^{\mp \frac{1}{2}} \rightarrow 0} \mathcal{I}_\alpha^Y$$

$$\lim_{q \rightarrow t^{\pm \frac{1}{2}}} \mathcal{I}_\alpha^X = \lim_{q \rightarrow t^{\mp \frac{1}{2}}} \mathcal{I}_\alpha^Y$$

Proof?

- Exchanges perturbative and non-perturbative contributions



$T[SU(2)]$ **Example**

Self-mirror dual

$$\mathcal{I}^Y = \mathcal{I}_{1\text{-loop}} \mathcal{I}_{\text{Vortex}} = \frac{(qx; q)_{\infty}}{(tqx; q)_{\infty}} \sum_{k \geq 0} \xi^k \frac{(tqx; q)_k}{(qx; q)_k}$$

$$\mathcal{I}^X = \mathcal{I}_{1\text{-loop}} \mathcal{I}_{\text{Vortex}} = \frac{(qx; q)_{\infty}}{(x; q)_{\infty}} \sum_{k \geq 0} (t^{-1} q \xi)^k \frac{(tqx; q)_k}{(qx; q)_k}$$

$$\lim_{\text{Poincaré}} \mathcal{I}^Y = \mathcal{I}_{\text{Vortex}} = \sum_{k \geq 0} \xi^k \quad \quad \quad \lim_{\text{Poincaré}} \mathcal{I}^X = \mathcal{I}_{1\text{-loop}} = \frac{1}{1-x}$$

Generating function of handsaw quiver Poincaré polynomials [\[Nakajima\]](#)

Twisted index and Hilbert series

Twisted index on $\Sigma = S^2$ of 3d $\mathcal{N} = 4$ theories computes Hilbert series

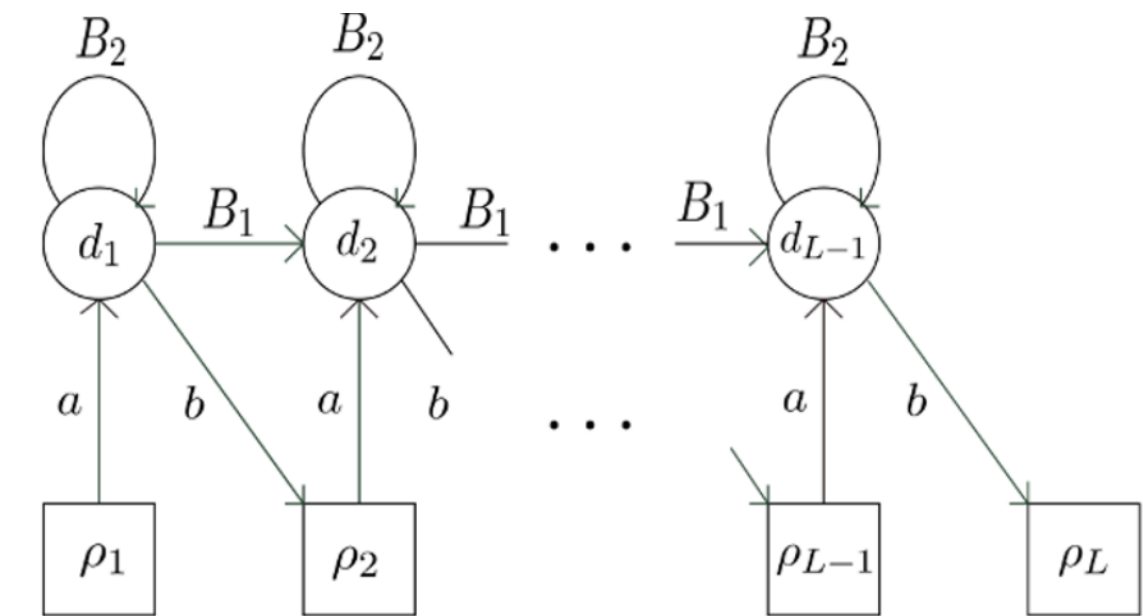
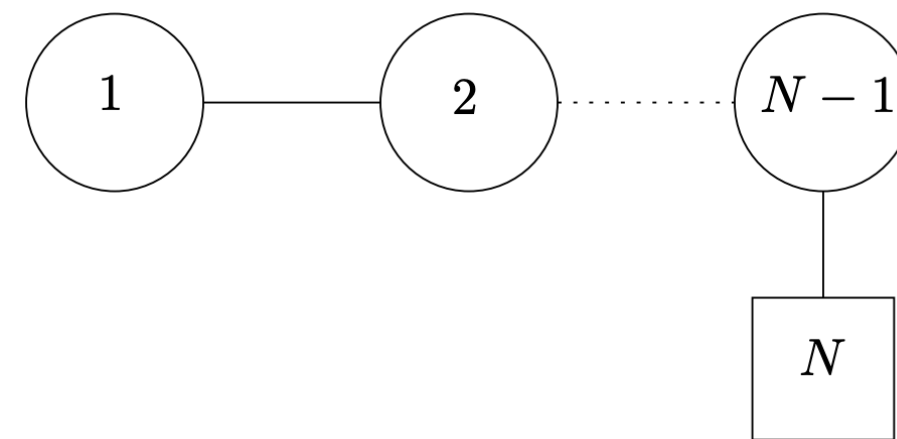
Example $T[SU(N)]$

Twisted index/Hilbert series factorises:

$$\mathcal{Z}_{S^2 \times_A S^1} = \sum_{\alpha \in S_N} \mathcal{Z}_{S^1 \times H^2}(q, t) \mathcal{Z}_{S^1 \times H^2}(q^{-1}, t)$$

Free to send $q \rightarrow 0$

$$\begin{aligned} \mathcal{Z}_{S^1 \times_A S^1} &= \sum_{\alpha \in S_N} \prod_{i < j} \frac{1}{1 - tx_i/x_j} \prod_{i > j} \frac{1}{1 - tx_i/x_j} \\ &= \prod_{i,j=1}^N \frac{1}{1 - tx_i/x_j} = \text{H.S. } (T[SU(N)]) \end{aligned}$$



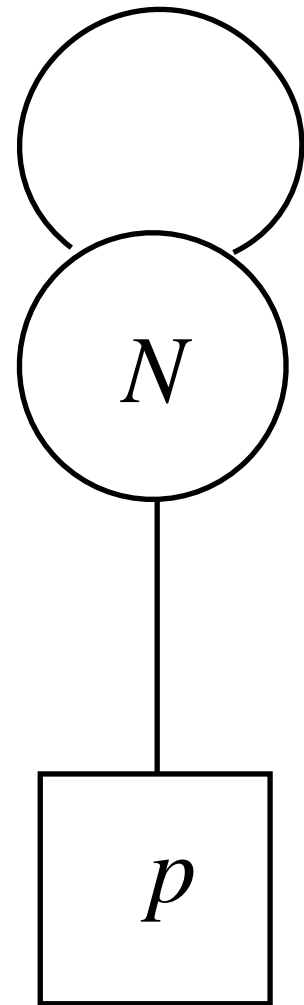
1-loop contributions only i.e. Poincaré polynomials

$$\sum_{\vec{d}} x^{\vec{d}} P_t(\mathfrak{Q}_{\alpha}^{\vec{d}})$$

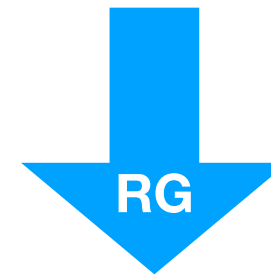
Recap so far

- Twisted index computes Hilbert series
- Factorising twisted index gives formula for Hilbert series in terms of Poincaré polynomial of quasi-map space
- 3d mirror symmetry provides generating function of Poincaré polynomials
- Concrete for $T_\rho[SU(N)]$ theories — Handsaw quivers
- Will now discuss more complicated, “speculative” example

3d ADHM Example



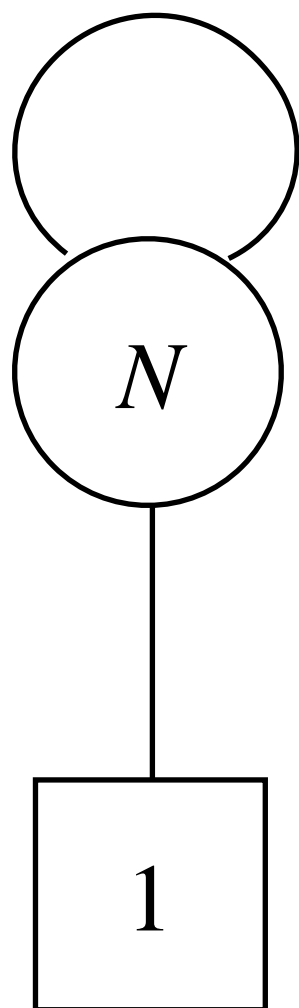
Worldvolume theory N $D2$ branes on single $D6$ in type IIA $p = 1$



N $M2$ branes $\longleftrightarrow$ SUGRA $AdS_4 \times S^7$

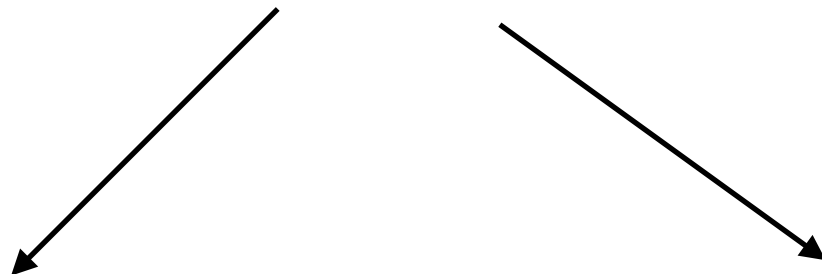
BH entropy in Cardy limit [Choi and Hwang] from vortex partition functions

$$\lim_{q \rightarrow 1} \oint e^{\frac{1}{\log(q)} \mathcal{W}} = H^* \sim e^{\frac{1}{\log(q)} N^{\frac{3}{2}} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}}$$



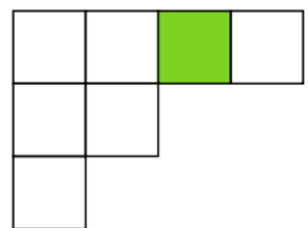
- Mirror self dual

Torus action on $\text{Hilb}^N(\mathbb{C}^2)$



- $\mathcal{M}_H = \text{Hilb}^N(\mathbb{C}^2)$ with $G_H = U(1)$ and $R_H - R_C = U(1)$

- Vacua labelled by partitions $|\lambda| = N$



$$E_{k,t} = \sum_{a=1}^{N_k} \frac{\prod_b w_{k,a} - w_{k-1,b} - \epsilon_2}{\prod_{b \neq a} (w_{k,a} - w_{k,b})} w_{k,a}^t v_{k,a}$$

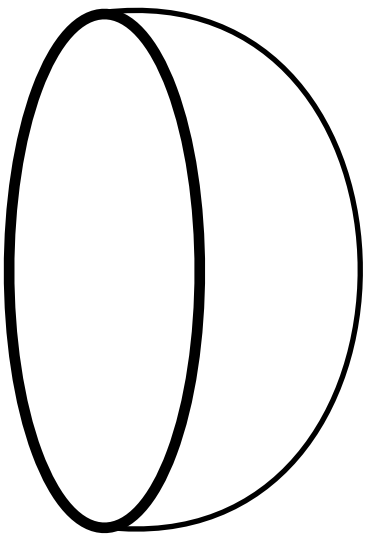
- $\hat{\mathbb{C}}[\mathcal{M}_C]$ described by [Nakajima and Kodera]

$$F_{k,t} = \sum_{a=1}^N \frac{\prod_b w_{k,a} - w_{k+1,b} + \epsilon_2}{\prod_{b \neq a} (w_{k,a} - w_{k,b})} v_{k,a}^{-1} w_{k,a}^{t+\delta_{0,k}}$$

- Cyclotomic rational Cherednik algebra
- $\mathbb{C}[\mathcal{M}_H]$ gauge invariant polynomials in (A, B, I, J)

$$\text{QM}_\lambda^d(\mathbb{P}^1 \rightarrow \text{Hilb}^N(\mathbb{C}^2)) \qquad \mathcal{Z}_{\text{Vortex}} = \sum_d \zeta^d \chi(\hat{\mathcal{O}}_{\text{Vir.}})$$

ADHM hemisphere partition function

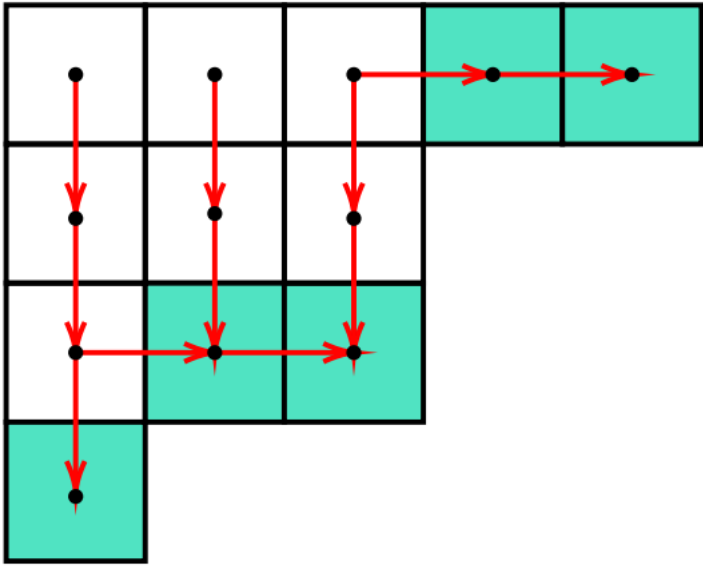


$$\mathcal{Z}_{S^1 \times D}^\lambda = \mathcal{Z}_{\text{Classical}}^\lambda \mathcal{Z}_{1\text{-loop}}^\lambda \mathcal{Z}_{\text{Vortex}}^\lambda$$

New

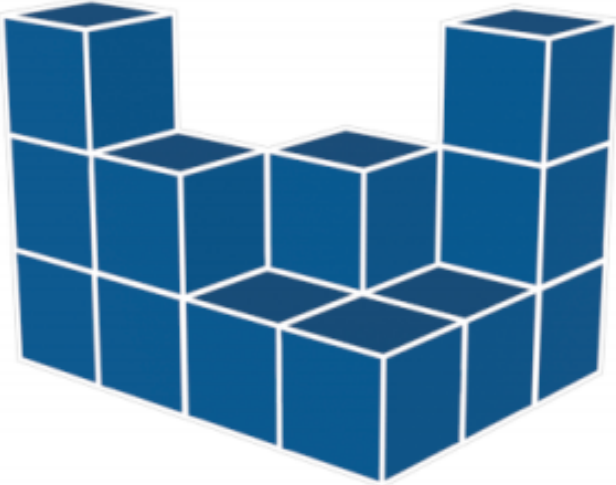
$$\mathcal{Z}_{\text{Classical}}^\lambda = e^{-\left[\sum_{s \in \lambda} c(s)\right] \frac{\log \zeta \log z}{\log q}} e^{\left[\sum_{s \in \lambda} h(s)\right] \frac{\log v \log z}{\log q}} e^{\left[\sum_{s \in \lambda} h(s)\right] \frac{\log u \log \zeta}{\log q}} e^{-\left[\sum_{s \in \lambda} c(s)\right] \frac{\log u \log v}{\log q}}$$

$$\mathcal{Z}_{1\text{-loop}}^\lambda = \prod_{s \in \lambda} \frac{(qz^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)-1}; q)_\infty}{(z^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)+1}; q)_\infty}$$



Familiar-ish

$$\mathcal{Z}_{\text{Vortex}}^\lambda = \sum_{\pi \in \text{RPP}(\lambda)} \left(\zeta t^{\frac{1}{2}} q^{-\frac{1}{4}}\right)^{|\pi|} \prod_{s \in \lambda} \frac{(u^2 v_s^{-1}; q)_{-\pi_s}}{(q v_s^{-1}; q)_{-\pi_s}} \prod_{\substack{s, t \in \lambda \\ s \neq t}} \frac{\left(qu^{-2} \frac{v_t}{v_s}; q\right)_{\pi_t - \pi_s}}{\left(\frac{v_t}{v_s}; q\right)_{\pi_t - \pi_s}} \frac{\left(zu \frac{v_t}{v_s}; q\right)_{\pi_t - \pi_s}}{\left(qzu^{-1} \frac{v_t}{v_s}; q\right)_{\pi_t - \pi_s}}$$



$$u = t^{\frac{1}{2}} q^{\frac{1}{4}}, \quad v = t^{-\frac{1}{2}} q^{\frac{1}{4}} \qquad \text{Fuses exactly!}$$

Verma character limits

$$\mathcal{Z}_{\text{Vortex}}^\lambda = \sum_{\pi \in \text{RPP}(\lambda)} \left(\zeta t^{\frac{1}{2}} q^{-\frac{1}{4}} \right)^{|\pi|} \prod_{s \in \lambda} \frac{(u^2 v_s^{-1}; q)_{-\pi_s}}{(q v_s^{-1}; q)_{-\pi_s}} \prod_{\substack{s, t \in \lambda \\ s \neq t}} \frac{\left(q u^{-2} \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}{\left(\frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}} \frac{\left(z u \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}{\left(q z u^{-1} \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}$$

$$\mathcal{Z}_{1\text{-loop}}^\lambda = \prod_{s \in \lambda} \frac{(q z^{a_\lambda(s) + l_\lambda(s) + 1} u^{-a_\lambda(s) + l_\lambda(s) - 1}; q)_\infty}{(z^{a_\lambda(s) + l_\lambda(s) + 1} u^{-a_\lambda(s) + l_\lambda(s) + 1}; q)_\infty}$$

Verma limit

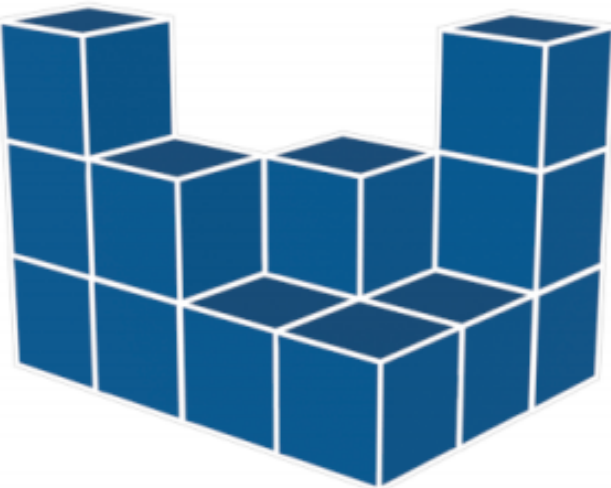
Mirror Verma limit

$$\sum_{\pi \in \text{RPP}(\lambda)} \zeta^{|\pi|}$$

$$=$$

$$\prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s) + l_\lambda(s) + 1}}$$

Fixed points on QM space



Generating function

Representation theory

- Verma modules of $\hat{\mathbb{C}}[\mathcal{M}_C]$?
- π boundary operators $\longleftrightarrow$ Fixed points on $\text{QM}_\lambda^d(\text{Hilb}^N(\mathbb{C}^2))$
- Action of $\hat{\mathbb{C}}[\mathcal{M}_C]$ on RPPs?

$$\begin{array}{ccc}
 \mathcal{Z}_{\text{Classical}} & & \mathcal{Z}_{\text{Vortex}} \\
 \searrow & & \swarrow \\
 \chi_\lambda(\hat{\mathbb{C}}[\mathcal{M}_C]) = & e^{\frac{\xi m}{2\beta} \sum_{s \in \lambda} c_\lambda(s) + \frac{\xi}{2} \sum_{s \in \lambda} h_\lambda(s)} & \sum_{\pi \in \text{RPP}(\lambda)} \zeta^{|\pi|}
 \end{array}$$

Poincaré polynomial limit

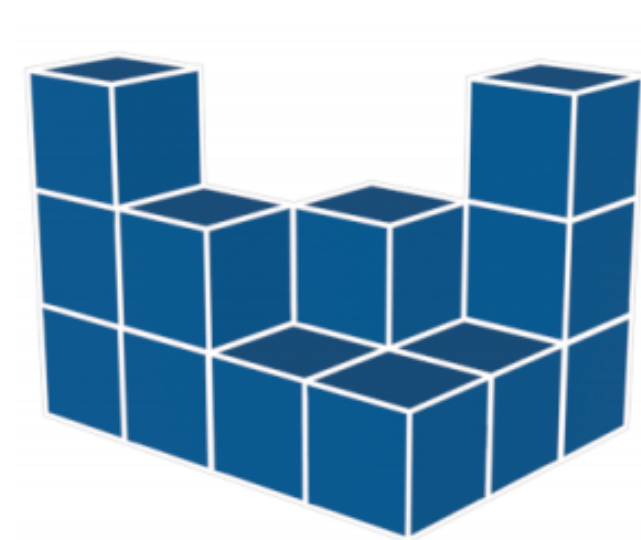
$$\mathcal{Z}_{\text{Vortex}}^\lambda = \sum_{\pi \in \text{RPP}(\lambda)} \left(\zeta t^{\frac{1}{2}} q^{-\frac{1}{4}}\right)^{|\pi|} \prod_{s \in \lambda} \frac{(u^2 v_s^{-1}; q)_{-\pi_s}}{(q v_s^{-1}; q)_{-\pi_s}} \prod_{\substack{s, t \in \lambda \\ s \neq t}} \frac{\left(q u^{-2} \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}{\left(\frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}} \frac{\left(z u \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}{\left(q z u^{-1} \frac{v_t}{v_s}; q \right)_{\pi_t - \pi_s}}$$

$$\mathcal{Z}_{1\text{-loop}}^\lambda = \prod_{s \in \lambda} \frac{(q z^{a_\lambda(s) + l_\lambda(s) + 1} u^{-a_\lambda(s) + l_\lambda(s) - 1}; q)_\infty}{(z^{a_\lambda(s) + l_\lambda(s) + 1} u^{-a_\lambda(s) + l_\lambda(s) + 1}; q)_\infty}$$



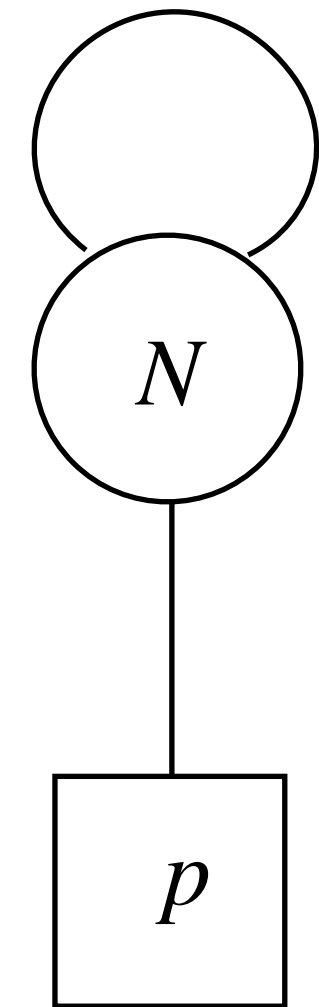
$$\sum_{\pi \in \text{RPP}(\lambda)} t^{\frac{1}{2}(\text{ht}'(\pi) - \text{ht}(\pi) + b(\pi))} z^{|\pi|} \quad = \quad \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s) + l_\lambda(s) + 1} t^{\frac{1}{2}(-a_\lambda(s) + l_\lambda(s) + 1)}}$$

Refined generating function of RPPs



Neumann boundary condition

- Throughout we have been using particular Dirichlet boundary conditions
- Neumann boundary condition is expected to yield simple modules
[Bullimore, Dimofte, Gaiotto and Hilburn]
- Consider $p = N_f > 1$. I.e. general Jordan quiver
- $\mathcal{M}_H = \mathcal{M}_{N,p}$ instanton moduli space $G_H = SU(p)$



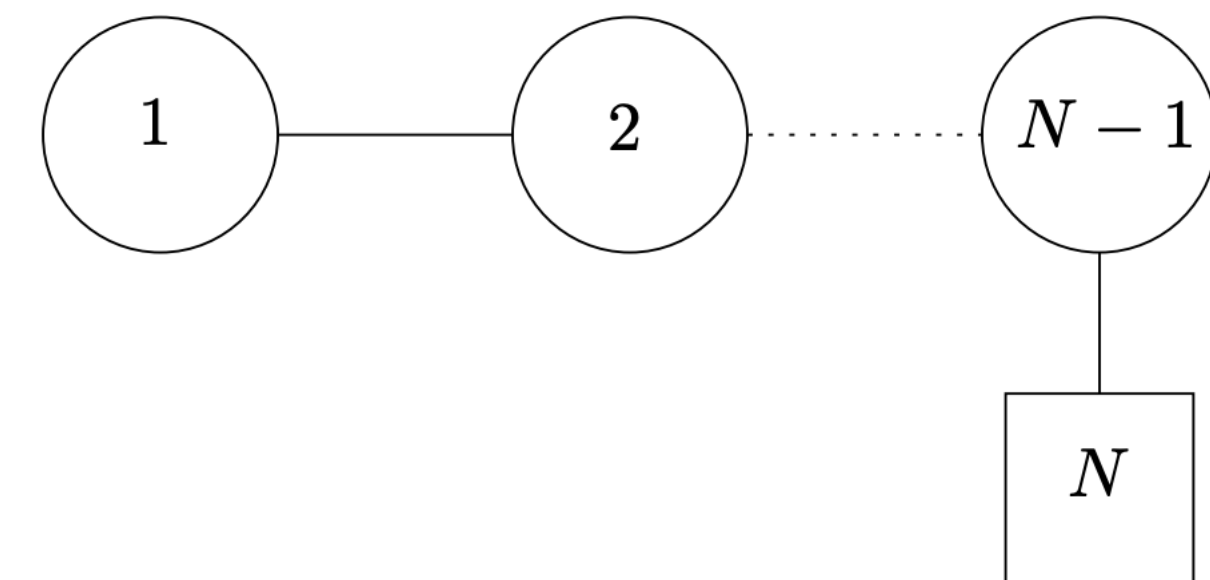
$T[SU(N)]$ analogy

- Neumann boundary condition is computed via contour integral

$$\mathcal{Z}_{S^1 \times D}[z_i, \zeta_i; q, t] = \oint_{\Gamma} \prod_{a=1}^N \prod_{i=1}^a \frac{dx_a^{(i)}}{x_i^{(a)}} e^{\log(x_i^{(a)}) \log(\zeta_i^{(a)})} \prod_{a=1}^{N-1} \frac{\prod_{i \neq j}^a (x_j^{(a)} / x_i^{(a)}; q)_{\infty}}{\prod_{i,j}^N (tq x_j^{(a)} / x_i^{(a)}; q)_{\infty}} \prod_{a=1}^{N-1} \prod_{i=1}^a \prod_{j=1}^{a+1} \frac{(tq x_j^{(a+1)} / x_i^{(a)}; q)_{\infty}}{(x_j^{(a+1)} / x_i^{(a)}; q)_{\infty}}$$

Integral form of Macdonald polynomial

$\downarrow$ Verma limit
 S_{λ}



Borel-Weil-Bott — compact Lagrangian core is complete flag

Equivalent for ADHM

$$\mathcal{Z}_{S^1 \times D}^B = \frac{1}{N!} \oint_{\text{JK}} \prod_{a=1}^N \frac{ds_a}{2\pi i s_a} s_a^{-\zeta} \frac{\prod_{a \neq b}^N (s_a s_b^{-1}; q)_\infty}{\prod_{a,b=1}^N (s_a s_b^{-1} t^{-1} q; q)_\infty} \prod_{a,b=1}^N \frac{(s_a s_b^{-1} z t^{-\frac{1}{2}} q; q)_\infty}{(s_a s_b^{-1} z t^{\frac{1}{2}} q; q)_\infty} \prod_{a=1}^N \prod_{i=1}^p \frac{(s_a x_i t^{-\frac{1}{2}} q; q)_\infty}{(s_a x_i t^{\frac{1}{2}} q; q)_\infty}.$$

Choice of $\mathcal{B}$

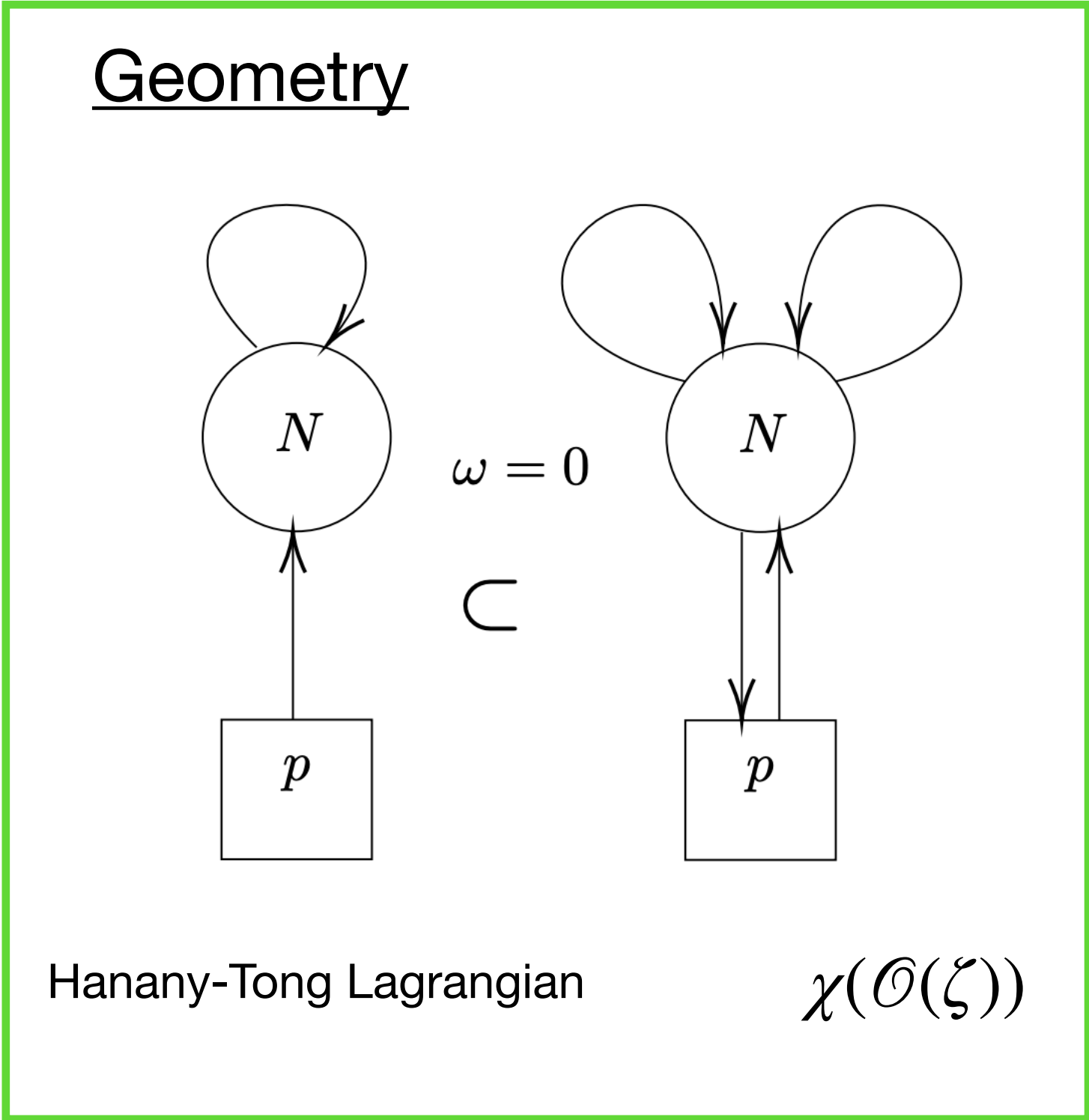
Verma limit

$$\mathcal{Z}_{N,p}(\mathcal{B}) = \frac{1}{N!} \oint_{(S^1)^N} \prod_{a=1}^N \frac{ds_a}{2\pi i s_a} s_a^{-\zeta} \frac{\prod_{a \neq b}^N (1 - s_a s_b^{-1})}{\prod_{a,b=1}^N (1 - z s_a s_b^{-1})} \prod_{a=1}^N \prod_{i=1}^p \frac{1}{1 - x_i s_a}$$

$$\mathcal{Z}_{N,p}(\mathcal{B}) = Q'_{(\zeta)^N}(x_1, \dots, x_p; z)$$

Milne polynomial $q \rightarrow 0$ of Haiman-Garcia Macdonald polynomial

Character of KR module / XXZ spin chain partition function



Simple module of Nakajima and Kodera $\hat{\mathbb{C}}[\mathcal{M}_C]$?

Refined Topological Vertex

- Quasi-maps to $\text{Hilb}^N(\mathbb{C}^2)$ should be related to vertex
- $q \rightarrow 0$ limit coincides with the refined topological vertex

$$\lim_{q \rightarrow 0} \mathcal{Z}_{S^1 \times D}^{A, \lambda} = t_1^{\frac{1}{4} \|\lambda\|^2} t_2^{-\frac{1}{4} \|\lambda^\vee\|^2} \prod_{s \in \lambda} \frac{1}{1 - t_1^{l_\lambda(s)+1} t_2^{-a_\lambda(s)}}$$

1-loop piece

Classical piece = framing factors

$$C_{\emptyset, \emptyset, \lambda}^{(\text{IKV})}(t = t_2^{-1}, q = t_1)$$

Twisted index gluing

$$\sum_{\substack{\lambda \\ |\lambda|=N}} \lim_{q \rightarrow 0} \mathcal{Z}_{\text{Classical}}^{B,\lambda} \bar{\mathcal{Z}}_{\text{Classical}}^{B,\lambda} \mathcal{Z}_{\text{1-loop}}^{B,\lambda} \bar{\mathcal{Z}}_{\text{1-loop}}^{B,\lambda}$$

- Twisted index = Hilbert series of $\text{Hilb}^N(\mathbb{C}^2)$
- Identify the twisted index gluing with the gluing of vertices.

Poincaré polynomials

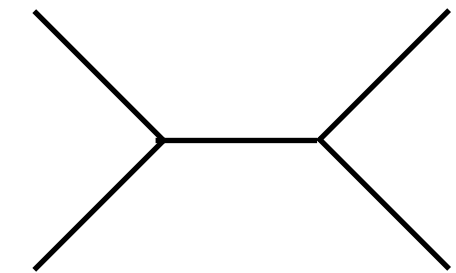
$$\lim_{q \rightarrow 0} \mathcal{Z}_{\text{1-loop}}^{B,\lambda}(z, \zeta; q, t) = \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)+1)}},$$

$$\lim_{q \rightarrow 0} \mathcal{Z}_{\text{1-loop}}^{B,\lambda}(z, \zeta; q^{-1}, t) = \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)-1)}},$$

$$\lim_{q \rightarrow 0} \mathcal{Z}_{\text{Classical}}^{B,\lambda}(z, \zeta; q, t) = z^{\frac{1}{2} \sum_{s \in \lambda} h_\lambda(s)} t^{-\frac{1}{4} \sum_{s \in \lambda} c_\lambda(s)},$$

$$\lim_{q \rightarrow 0} \mathcal{Z}_{\text{Classical}}^{B,\lambda}(z, \zeta; q^{-1}, t) = z^{\frac{1}{2} \sum_{s \in \lambda} h_\lambda(s)} t^{-\frac{1}{4} \sum_{s \in \lambda} c_\lambda(s)}.$$

$$\mathcal{Z}_{\text{Vortex}}^{A,\lambda} = V_{\text{PT}}^{\emptyset, \emptyset, \lambda}$$



Interpret as conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ amplitude

Independence of q ...

$$= \sum_{\substack{\lambda \\ |\lambda|=N}} \prod_{s \in \lambda} \frac{z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s))}}{\left(1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)+1)}\right) \left(1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)-1)}\right)}$$

← Nekrasov's partition function

Summary

- Hemisphere partition functions realise factorisation
- Verma character formulae for $\mathcal{M}_3$ partition functions
 - Geometric interpretation
- Twisted index, Hilbert series and Poincaré polynomials
- Detailed study of ADHM example
 - Connections to topological vertex

Further directions

- Quasi-map interpretation of 1-loop contributions
- Mirror symmetry of boundary conditions
- Geometric interpretation of Cardy limit $\lim_{q \rightarrow 1} \mathcal{Z}_{S^1 \times D}^\lambda$
 - Phase with non-trivial scaling of $N = |\lambda|$ i.e. $\sim e^{N^{3/2}}$
 - Relevance of Hanany-Tong lagrangian and simple modules?

Thanks